TAIL ASYMPTOTICS FOR SHEPP-STATISTICS OF BROWNIAN MOTION IN $\mathbb{R}^d$

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Abstract. Let $\boldsymbol{X}(t)$, $t \in \mathbb{R}$, be a d-dimensional vector-valued Brownian motion, $d \geq 1$. For all $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ we derive exact asymptotics of

$$\mathbb{P}\{\boldsymbol{X}(t+s) - \boldsymbol{X}(t) > u\boldsymbol{b} \text{ for some } t \in [0,T],\ s \in [0,1]\} \quad \text{as } u \to \infty,$$

that is the asymptotical behavior of tail distribution of vector-valued analog of Shepp-statistics for $\boldsymbol{X}$; we cover not only the case of a fixed time-horizon $T > 0$ but also cases where $T \to 0$ or $T \to \infty$. Results for high level excursion probabilities of vector-valued processes are rare in the literature, with currently no available approach suitable for our problem. Our proof exploits some distributional properties of vector-valued Brownian motion, and results from quadratic programming problems. As a by-product we derive a new inequality for the 'supremum' of vector-valued Brownian motions.

Key Words: Shepp-statistics; vector-valued Brownian motion; high level excursion probability; uniform double-sum method; markov property; quadratic programming problem.

1. Introduction

For $B(t)$, $t \in \mathbb{R}$, a standard Brownian motion define *the Shepp statistics*

$$m(t) = \sup_{s \in [0,1]} [B(t+s) - B(t)], \quad t \geq 0.$$

In numerous theoretical problems and applications motivated by the fact that the Brownian motion is a natural limit process, investigation of $M(T) := \sup_{t \in [0,T]} m(t)$ is of particular interest, see e.g., [3, 6, 11, 14, 15, 18–20]. The asymptotics of high level excursion probability of $M(t)$, $t \geq 0$, was first derived in [23], giving

$$\mathbb{P}\{M(T) > u\} \sim T\mathcal{H}^* u\varphi(u) \quad \text{as } u \to \infty, \tag{1}$$

where φ is the density function of $B(1)$ and the positive constant $\mathcal{H}^*$ is given by

$$\mathcal{H}^* = \lim_{\lambda \to \infty} \lim_{\tau \to \infty} \frac{1}{\tau} \mathbb{E}\left\{ \sup_{s \in [0,\lambda], t \in [0,\tau]} e^{B(t+s+\tau) - B(t) - (\tau+\lambda)/2} \right\}.$$

Interestingly, $\mathcal{H}^*$ is not the classical Pickands constant, which commonly appears in the asymptotics of extremes of Gaussian processes; see, e.g., [4, 5, 12, 13, 16, 17] for definition, various representations and basic properties of Pickands constants.

In this contribution we shall investigate the high level excursion probability of vector-valued Shepp statistics. Let therefore $\boldsymbol{B}(t) = (B_1(t), \ldots, B_d(t))^\top$, $t \geq 0$, be a vector-valued random process with components which are mutually independent standard Brownian motions and define

$$\boldsymbol{Y}(t,s) = \boldsymbol{X}(t+s) - \boldsymbol{X}(t),$$

Date: July 2, 2019.

where $\boldsymbol{X}(t) = A\boldsymbol{B}(t)$ and A is a non-singular matrix of size $d \times d$. Using the fact that the correlation matrix of $\boldsymbol{X}(t)$ is Σt with $\Sigma := AA^\top$, and Σ is a symmetric positive definite matrix, one can find a symmetric positive definite matrix $\sqrt{\Sigma}$ such that $A\boldsymbol{B}(t)$ and $\sqrt{\Sigma}\boldsymbol{B}(t)$ have the same covariance structure. Thus without loss of generality, in the rest of the paper we assume that $A = \sqrt{\Sigma}$. We note in passing that all the vectors in $\mathbb{R}^d$ are denoted by bold symbols. Operations on vectors are component-wise.

Given a fixed $\boldsymbol{b} = (b_1, \dots, b_d)^\top$, in this contribution we shall derive the exact asymptotics of the following high level excursion probability

$$p(u) \quad := \quad \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \in [0,T],\ s \in [0,1]\right\} \quad \text{as } u \to \infty. \tag{2}$$

Since we are interested in the case that $p(u) \to 0$ as $u \to \infty$, in the rest of this paper we tacitly assume that $b_i > 0$ for some $1 \le i \le d$. One of important motivations to analyse (2) is the connection with the conjunction problem for Gaussian fields; see for example [21, 22]. The set of conjunctions $C_{T,u}$ with respect to some threshold u is defined as

$$C_{T,u} := \{(t,s) \in [0,T] \times [0,1] : \min_{1 \le i \le d} Y_i(t,s) > u\}.$$

One of the key characteristics of interest for $C_{T,u}$ is the probability that this set is non-empty, which is a special case of (2) with $b_i = 1$ for all $1 \le i \le d$.

There are very few contributions in the literature that are devoted to the study of extremes of vector-valued Gaussian processes. The principal reason for this is that the Slepian inequality is not valid for general vector-valued Gaussian processes, and thus so far no general methodology exists for the study of the excursion probabilities in the vector-valued setup.

For less difficult problems, such as the derivation of the logarithmic asymptotics of $p(u)$, several results for a large class of vector-valued Gaussian processes can be found in [10].

Finer asymptotic approximations are indeed available in the literature, however their proofs have significant gaps (due to lack of Slepian inequality mentioned above). For instance the approximations of high excursion probabilities derived in [1] have gaps related to the lack of the proof of the uniformity of several results with respect the the summand that leads to the final asymptotics. We refer also to [2, 8] which deal with tail approximation of supremum of order statistics of vector-valued Gaussian processes, where only the case of independent components is considered, for which an extension of Slepian inequality, that is Gordon inequality is available.

The recent contribution [7] considers the infinite time ruin probability related to $\boldsymbol{X}(t)$, $t \in [0,\infty)$, with linear drift. As shown therein, rigorous proofs require subtle uniform approximations, which are complex and quite specific to the Brownian motion case. Additional complexity of the problem considered in the present contribution relates to the fact that (2) concerns random fields, whereas [7] is dedicated to random processes.

As in the one-dimensional case analyzed in [23], in order to investigate the high level excursion probabilities of Shepp-Statistics, the following three properties are essential:

i) for any fixed s the vector-valued Gaussian family (indexed by s) $\boldsymbol{W}_s(t) := \boldsymbol{Y}(t,s)$, $t \in [0,T]$, is stationary;
ii) for any fixed t the variance of the vector-valued Gaussian family (indexed by t) $\boldsymbol{V}_t(s) := \boldsymbol{Y}(t,s)$, $s \in [0,1]$, attains its unique maximum on $[0,1]$ at the right-end point $s = 1$;

iii) the independence of increments of $\boldsymbol{X}$.

In order to derive the asymptotics of (2) we apply the *uniform double-sum* technique. This method was originally developed for studying extremes of centered non-stationary Gaussian processes and fields, e.g. [16, 17], see also the recent contribution [9] for the role of uniformity and extensions to general functionals of Gaussian random processes and fields. Note in passing that Pickands approach [16], also often referred to as the *double-sum* technique, is significantly different from the uniform-double sum technique here (or from that developed by Piterbarg, see [17]); we do not use discretisation approach, but apply directly continuous mapping theorem to some conditional process. Importantly, our approximations are uniform with respect to the small intervals we consider. In the case of processes, this can be shown in the non-stationary case by using Slepian inequality. In the vector-valued case, such inequality is in general not valid. Our uniform approximations are shown by utilising mainly the self-similarity and the independence of increments property of Brownian motion.

In the classical approach developed by Piterbarg for the investigation of extremes of Gaussian random fields, the Slepian inequality is in fact used twice, once for the approximation of the so-called single sum, and then for the negligibility of the double-sum; see the recent contribution [9] for more details and critical issues regarding uniformity. Again, in the vector-valued case, the negligibility of the double-sum, which has to be approximated uniformly, cannot be shown by the standard method of Piterbarg (recall the lack of Slepian inequality).

We solve this problem by using the Markov property, the self-similarity property, the independence and the stationarity of increments as well as the continuity of the sample paths of the Brownian motion.

In the setup of this paper, the stationarity of $\boldsymbol{W}_s(t)$, $t \in [0, T]$, for fixed s is important, since the extremes of stationary processes are well-understood. There is however a hidden and subtle difficulty here since all these stationary vector-valued Gaussian processes are indexed by s. So when using results for stationary processes, the uniformity with respect to s is crucial and cannot be neglected. This fact has not been explicitly addressed in [23] when dealing with the negligibility of some terms that appear in the lower bound approximation.

In this paper we use a direct approach to show the negligibility of analogs of those terms. Note that our approach is valid also for the case $d = 1$.

An interesting by-product of our investigation is the following elegant result which is new even for the case $d = 1$, $m > 1$.

Theorem 1.1. *Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ be m independent copies of $\boldsymbol{X}$. Then for all $\boldsymbol{b} \in \mathbb{R}^d$ and $T_1, \ldots, T_m > 0$, we have*

$$\mathbb{P}\left\{\sum_{i=1}^{m} \boldsymbol{X}_i(t_i) \geq \boldsymbol{b} \text{ for some } t_1 \in [0, T_1], \ldots, t_m \in [0, T_m]\right\} \;\leq\; \frac{\mathbb{P}\{\boldsymbol{X}(T_1 + \ldots + T_m) \geq \boldsymbol{b}\}}{(\mathbb{P}\{\boldsymbol{X}(1) \geq \boldsymbol{0}\})^m}.$$

This inequality may be viewed at as a multi-dimensional analogue of the well-known distribution equality in dimension 1,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} B(t) > b\right\} \;=\; 2\mathbb{P}\{B(T) > b\} \;=\; \frac{\mathbb{P}\{B(T) > b\}}{\mathbb{P}\{B(1) > 0\}}.$$

As follows from the proof of Theorem 1.1 in Section 3.2, the last equality does not hold in higher dimensions.

Organisation of the rest of the paper. In Section 2 we introduce some useful notation and a related quadratic programming problem which determines the exponential part of the asymptotics. The main result of this contribution is given in Theorem 2.1. Section 3 is devoted to the proof of Theorem 2.1, while in the Appendix we present a lemma that deals with properties of the unique solution to some quadratic programming problem.

2. Main result

Before proceeding to the main result of this contribution, let us begin with the analysis of some quadratic programming problem, whose solution determines the exponential part of the asymptotics of (2).
Hereinafter $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty, 0]^d$. Let $\widetilde{\boldsymbol{b}} = (\tilde{b}_1, \dots, \tilde{b}_d)$ be a unique solution to the quadratic programming problem

$$\Pi_\Sigma(\boldsymbol{b}): \quad \text{minimise the quadratic form } \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \text{ for } \boldsymbol{x} \in \mathbb{R}^d \text{ under the constraints } \boldsymbol{x} \geq \boldsymbol{b}. \tag{3}$$

Naturally, the minimal attained value of $\Pi_\Sigma(\boldsymbol{b})$ is $\widetilde{\boldsymbol{b}}^\top \Sigma^{-1} \widetilde{\boldsymbol{b}}$. Equivalently, we can rewrite the problem $\Pi_\Sigma(\boldsymbol{b})$ as

$$A^{-1}\widetilde{\boldsymbol{b}} \text{ minimises } \boldsymbol{y}^\top \boldsymbol{y} \text{ for } \boldsymbol{y} \in \mathbb{R}^d \text{ under the constraints } \boldsymbol{y} \in A^{-1}\boldsymbol{b} + V, \tag{4}$$

where $V = A^{-1}(\mathbb{R}^+)^d$ is a convex cone. Let

$$L := \{1 \leq i \leq d : b_i = \tilde{b}_i\} \tag{5}$$

and further let $I \subseteq L$ be the minimal index set such that $\boldsymbol{b}_I = \widetilde{\boldsymbol{b}}_I$ and further

$$\boldsymbol{b}_I^\top \Sigma_{II}^{-1} \boldsymbol{b}_I \; = \; \boldsymbol{b}_L^\top \Sigma_{LL}^{-1} \boldsymbol{b}_L \; = \; \widetilde{\boldsymbol{b}}^\top \Sigma^{-1} \widetilde{\boldsymbol{b}}. \tag{6}$$

In our notation $\boldsymbol{b}_I$ is the subvector of $\boldsymbol{b}$ indexed by I, and similarly Σ_{II} is the submatrix of Σ with rows and columns with indices from I. For notational simplicity we write Σ_{II}^{-1} instead of $(\Sigma_{II})^{-1}$.
In view of Lemma A.1 in the Appendix, there may be a dimension reduction phenomenon which happens if the non-empty index set I has cardinality less than d, or equivalently $J := \{1, \dots, d\} \setminus I$ is non-empty. In particular, for the case $d = 2$ and Σ being a correlation matrix with off-diagonal entries equal to $\rho \in (-1, 1)$, that is

$$\Sigma \;=\; \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \qquad \Sigma^{-1} \;=\; \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix},$$

we have for all $\boldsymbol{x} \in \mathbb{R}^2$

$$g(\boldsymbol{x}) := \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} = (x_1^2 - 2\rho x_1 x_2 + x_2^2)/(1-\rho^2).$$

For $\boldsymbol{b} = (1, \rho)^\top$ we have that $g(\boldsymbol{b}) = 1$, $g'_{x_1}(\boldsymbol{b}) = 2$ and $g'_{x_2}(\boldsymbol{b}) = 0$, so the vector $\boldsymbol{e}_1 = (1, 0)$ is perpendicular to the ellipse $g(\boldsymbol{x}) = 1$; see Figure 1.

In this example $\widetilde{\boldsymbol{b}} = \boldsymbol{b}$, $L = \{1, 2\}$, $I = \{1\}$ and

$$\widetilde{\boldsymbol{b}}_I^\top \Sigma_{II}^{-1} \widetilde{\boldsymbol{b}}_I = \boldsymbol{b}^\top \Sigma^{-1} \boldsymbol{b} = 1 > 0.$$

If $\boldsymbol{b} = (1, a)^\top$ for some $a < \rho$, then $L = I = \{1\}$.

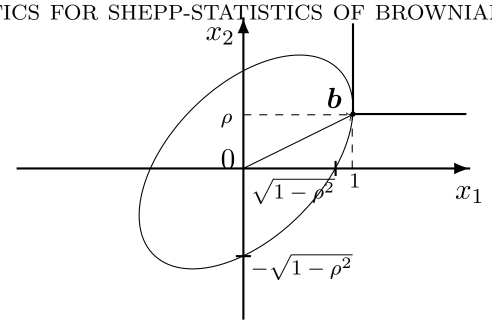

Figure 1.

Following Lemma A.1 (and because $\Sigma^{-1}\boldsymbol{b}$ is collinear with the gradient of the quadratic form $\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}$ at point $\boldsymbol{b}$), if $\Sigma^{-1}\boldsymbol{b} > \boldsymbol{0}$, then

$$I = \{1, \dots, d\}.$$

However, if $|I| < d$, then we observe a dimension reduction phenomenon, i.e., the asymptotics of (2) (up to a constant) is only determined by the components of $\boldsymbol{X}$ with indices in I.
Next, we introduce a key constant, which appears in the exact asymptotics of (2). For the introduced above index sets I, J, L, matrix Σ and vector $\boldsymbol{b}$ we define

$$\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda) \;:=\; \lim_{\tau\to\infty} \tau^{-1}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda), \tag{7}$$

where

$$\begin{aligned}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda) \;:=\; & e^{-\frac{\tau+\lambda}{2}\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}}\int_{\mathbb{R}^d} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I-\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J/2}\\ & \times\mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau)-\boldsymbol{X}_I(t) > \boldsymbol{x}_I,\ x_i<0, \forall i\in L\setminus I, \quad \text{for some } t\le\tau,\ s\le\lambda\right\} \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{8}$$

The pre-factor in our main result is the following constant

$$\mathcal{H}_{\Sigma,\boldsymbol{b}} \;:=\; \lim_{\lambda\to\infty}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda),$$

which by Lemma 3.6 (see Section 3) is well-defined, finite and positive. Note in passing that the aforementioned lemma also proves that $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda)$ is positive and finite for some $\lambda > 0$.
Hereafter, $\varphi_\Sigma(\cdot)$ stands for the density function of an $\mathcal{N}(\boldsymbol{0},\Sigma)$ random vector. The following theorem constitutes the main result of this paper.

Theorem 2.1. *For any $\boldsymbol{b}\in\mathbb{R}^d\setminus(-\infty,0]^d$ we have*

$$\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\in[0,T],\ s\in[0,1]\right\} \;\sim\; T\mathcal{H}_{\Sigma,\boldsymbol{b}}u^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}}), \tag{9}$$

as $u\to\infty$, uniformly for all $T := T(u)$ such that $\lim_{u\to\infty} Tu^2 = \infty$ and $\lim_{u\to\infty} Tu^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}}) = 0$.

As follows from considerations in the next section, the most probable path leading to the Shepp statistics being greater than $u\boldsymbol{b}$ for a large u is roughly speaking such that starting at some time t_0 between 0 and T the trajectory goes in direction $\boldsymbol{b}$ at a speed $u\boldsymbol{b}$ during time 1 approximately; the contribution from different values of t_0 is the same.

3. Proof of Theorem 2.1

In order to make the proof of Theorem 2.1 more transparent we divide it into several parts. We start with calculation of the tail probability for 'short' intervals for t close to zero and s close to 1, see Section 3.1, which is the key ingredient for further analysis. In the second step we derive the following upper bound (see Section 3.2):

$$\mathbb{P}\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\in[0,T],\ s\in[0,1]\} \ \le\ (T+o(1))u^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\mathcal{H}_{\Sigma,\boldsymbol{b}} \tag{10}$$

as $u\to\infty$ uniformly for all $T=T(u)$ such that $\lim_{u\to\infty} Tu^2=\infty$. Then, in Section 3.3 we prove the lower bound counterpart, namely that for all large u

$$\mathbb{P}\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\in[0,T],\ s\in[0,1]\} \ \ge\ (T+o(1))u^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\mathcal{H}_{\Sigma,\boldsymbol{b}} \tag{11}$$

uniformly for all T such that $\lim_{u\to\infty} Tu^2=\infty$ and the right hand side expression goes to zero. Finally, in Section 3.4, we check that $\mathcal{H}_{\Sigma,\boldsymbol{b}}\in(0,\infty)$. Altogether the above points establish the proof of Theorem 2.1.

3.1. Supremum for Small t and s Close to 1.

Lemma 3.1. *Let $\boldsymbol{b}\in\mathbb{R}^d\setminus(-\infty,0]^d$ be given and let $\widetilde{\boldsymbol{b}}$ be a unique solution to $\Pi_\Sigma(\boldsymbol{b})$ with I, J being the corresponding index sets. For all τ, λ positive we have, as $u\to\infty$,*

$$\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0\le t\le \tau/u^2,\ s\in[1-\lambda/u^2,1]\right\} \ \sim\ u^{-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda), \tag{12}$$

where $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)$ defined in (8) is positive and finite.

Proof of Lemma 3.1: For $u>0$ we set

$$\tau_u=\tau u^{-2},\quad \lambda_u=\lambda u^{-2},\quad p_u:=\sqrt{1-\lambda_u-\tau_u},\quad D_u:=[0,\tau_u]\times[0,\lambda_u].$$

By the independence of the increments of a standard Brownian motion B, for all $s\in[1-\lambda_u,1]$ and $t\in[0,\tau_u]$ we have

$$B(t+s)-B(t) \ = \ [B(t+s)-B(1-\lambda_u)]+[B(1-\lambda_u)-B(\tau_u)]+[B(\tau_u)-B(t)],$$

where the three differences on the right hand side are mutually independent, provided $\tau_u+\lambda_u\le 1$. Further, the stationarity of increments of B implies the equality in law

$$B(1-\lambda_u)-B(\tau_u) \ \stackrel{d}{=}\ p_u Z,$$

where Z is an $\mathcal{N}(0,1)$ random variable independent of B. Hence, again by the stationarity of increments

$$\begin{aligned} B(t+s)-B(t) \ &\stackrel{d}{=}\ p_uZ+[B(t+\tau_u+s-(1-\lambda_u))-B(\tau_u)]+[B(\tau_u)-B(t)] \\ &=\ p_uZ+B(t+\tau_u+s-(1-\lambda_u))-B(t). \end{aligned}$$

Consequently, by the independence of the components of $\boldsymbol{B}$, for all u large enough we have

$$\begin{aligned} \boldsymbol{Y}(t,s) \ &= \ A[\boldsymbol{B}(t+s)-\boldsymbol{B}(t)] \\ &\stackrel{d}{=}\ p_uA\boldsymbol{Z}+A\Big[\boldsymbol{B}(t+s+\tau_u-(1-\lambda_u))-\boldsymbol{B}(t)\Big],\quad (t,s-(1-\lambda_u))\in D_u, \end{aligned}$$

where $\boldsymbol{Z}$ has independent $\mathcal{N}(0,1)$ components being further independent of $\boldsymbol{B}$. Hence

$$\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le \tau_u,\ s \in [1-\lambda_u, 1]\right\}$$
$$= \mathbb{P}\left\{p_u A\boldsymbol{Z} + A[\boldsymbol{B}(t+s+\tau_u) - \boldsymbol{B}(t)] > u\boldsymbol{b} \text{ for some } (t,s) \in D_u\right\}.$$

Setting $\boldsymbol{Z}_u := p_u A\boldsymbol{Z}$ and denoting its covariance matrix by $\Sigma_u := p_u^2\Sigma$, we may further write

$$\begin{aligned}\mathbb{P}&\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le \tau_u,\ s \in [1-\lambda_u, 1]\right\}\\ &= \mathbb{P}\left\{\boldsymbol{Z}_u + A\big[\boldsymbol{B}(t+s+\tau_u) - \boldsymbol{B}(t)\big] > u\boldsymbol{b} \text{ for some } (t,s) \in D_u\right\}\\ &= \int_{\mathbb{R}^d} \varphi_{\Sigma_u}(-\boldsymbol{w})\mathbb{P}\left\{\boldsymbol{X}(t+s+\tau_u) - \boldsymbol{X}(t) > u\boldsymbol{b} + \boldsymbol{w} \text{ for some } (t,s) \in D_u\right\} \mathrm{d}\boldsymbol{w}.\end{aligned}$$

Let $\overline{\boldsymbol{u}} \in \mathbb{R}^d$ be a vector with coordinates $\overline{u}_i = u$ for all $i \in I$ and $\overline{u}_j = 1$ for all $j \in J$. Change of variables $\boldsymbol{w} = -u\widetilde{\boldsymbol{b}} + \boldsymbol{x}/\overline{\boldsymbol{u}}$, $\mathrm{d}\boldsymbol{w} = \mathrm{d}\boldsymbol{x}/u^{|I|}$, gives the following value of the last integral

$$\begin{aligned} u^{-|I|} &\int_{\mathbb{R}^d} \varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})\mathbb{P}\left\{\boldsymbol{X}(t+s+\tau_u) - \boldsymbol{X}(t) > u(\boldsymbol{b}-\widetilde{\boldsymbol{b}}) + \boldsymbol{x}/\overline{\boldsymbol{u}} \text{ for some } (t,s) \in D_u\right\} \mathrm{d}\boldsymbol{x}\\ &= u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) \int_{\mathbb{R}^d} e^{-\left((u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}}) - (u\widetilde{\boldsymbol{b}})^\top\Sigma^{-1}(u\widetilde{\boldsymbol{b}})\right)/2p_u^2}\\ &\quad \times \mathbb{P}\left\{\boldsymbol{X}(t+s+\tau) - \boldsymbol{X}(t) > u^2(\boldsymbol{b}-\widetilde{\boldsymbol{b}}) + u\boldsymbol{x}/\overline{\boldsymbol{u}} \text{ for some } t \le \tau,\ s \le \lambda\right\} \mathrm{d}\boldsymbol{x}.\end{aligned}$$

For all positive u, by the properties of the solution to $P_\Sigma(\boldsymbol{b})$–see Lemma A.1–we have

$$\begin{aligned}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(u\widetilde{\boldsymbol{b}} - \boldsymbol{x}/\overline{\boldsymbol{u}}) &= u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}} - 2u\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}}) + (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})\\ &= u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}} - 2u\boldsymbol{b}_I^\top\Sigma_{II}^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})_I + (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})\\ &= u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}} - 2\boldsymbol{b}_I^\top\Sigma_{II}^{-1}\boldsymbol{x}_I + (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}}),\end{aligned} \tag{13}$$

which implies

$$\begin{aligned}\mathbb{P}&\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le \tau_u,\ s \in [1-\lambda_u, 1]\right\}\\ &= u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) \int_{\mathbb{R}^d} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2 - (\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})/2p_u^2}\\ &\quad \times \mathbb{P}\left\{\boldsymbol{X}(t+s+\tau) - \boldsymbol{X}(t) > u^2(\boldsymbol{b}-\widetilde{\boldsymbol{b}}) + u\boldsymbol{x}/\overline{\boldsymbol{u}} \text{ for some } t \le \tau,\ s \le \lambda\right\} \mathrm{d}\boldsymbol{x}.\end{aligned} \tag{14}$$

For any $u > 0$ write

$$h_u(\boldsymbol{x}) := \mathbb{P}\left\{\boldsymbol{X}(t+s+\tau) - \boldsymbol{X}(t) > u^2(\boldsymbol{b}-\widetilde{\boldsymbol{b}}) + u\boldsymbol{x}/\overline{\boldsymbol{u}} \text{ for some } t \le \tau,\ s \le \lambda\right\}$$

for the probability under the integrand above. In view of $(\boldsymbol{b}-\widetilde{\boldsymbol{b}})_i = 0$ for all $i \in L$ (note that $I \subseteq L$ and I cannot be empty) and $(\boldsymbol{b}-\widetilde{\boldsymbol{b}})_i < 0$ for all $i \notin L$

$$\lim_{u\to\infty} h_u(\boldsymbol{x}) = \mathbf{1}_{\{x_i \le 0,\, \forall i \in L\setminus I\}}\mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau) - \boldsymbol{X}_I(t) > \boldsymbol{x}_I \text{ for some } t \le \tau,\ s \le \lambda\right\} =: h(\boldsymbol{x}).$$

Since $\lim_{u\to\infty} p_u = 1$, we have further

$$\begin{aligned}\mathbb{P}&\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le \tau_u,\ s \in [1-\lambda_u, 1]\right\}\\ &\sim u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) \int_{\mathbb{R}^d} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I - \boldsymbol{x}_J^\top\Sigma_{JJ}^{-1}\boldsymbol{x}_J/2} h(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \text{as } u \to \infty,\end{aligned} \tag{15}$$

where we have applied the dominated convergence theorem which is eligible because:

⋄ Firstly, the integrand in (14) is dominated by

$$\begin{aligned} & e^{\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2} e^{-(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top \Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})/2} \mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau)-\boldsymbol{X}_I(t) > \boldsymbol{x}_I \text{ for some } t\le\tau,\ s\le\lambda\right\} \\ & \quad\le\quad e^{-\delta\|\boldsymbol{x}_J\|^2} e^{\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2} \mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau)-\boldsymbol{X}_I(t) > \boldsymbol{x}_I \text{ for some } t\le\tau,\ s\le\lambda\right\} \end{aligned}$$

for some $\delta > 0$, because the matrix Σ^{-1} is positive definite.

⋄ Secondly, the function $e^{-\delta\|\boldsymbol{x}_J\|^2}$ is integrable with respect to $\boldsymbol{x}_J$ if J is non-empty, otherwise this term is missing.

⋄ Thirdly, the function

$$e^{\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2} \mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau)-\boldsymbol{X}_I(t) > \boldsymbol{x}_I \text{ for some } t\le\tau,\ s\le\lambda\right\} \tag{16}$$

is integrable with respect to $\boldsymbol{x}_I$ because, by Lemma A.1 we have

$$\boldsymbol{x}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I \quad=\quad \boldsymbol{x}_I^\top \boldsymbol{a}_I,$$

where $\boldsymbol{a}_I > \boldsymbol{0}_I$, and because of Piterbarg inequality–see [17, Theorem 8.1] (or the Borell-TIS inequality)–which ensures that

$$\begin{aligned} & \mathbb{P}\left\{\boldsymbol{X}_I(t+s+\tau)-\boldsymbol{X}_I(t) > \boldsymbol{x}_I \text{ for some } t\le\tau,\ s\le\lambda\right\} \\ & \quad\le\quad \mathbb{P}\left\{\sum_{i\in I: x_i>0}\big(X_i(t+s+\tau)-X_i(t)\big) > \sum_{i\in I: x_i>0} x_i \text{ for some } t\le\tau,\ s\le\lambda\right\} \\ & \quad\le\quad Ce^{-\varepsilon\left(\sum_{i\in I:x_i>0} x_i\right)^2} \end{aligned}$$

for some $C\in(0,\infty)$ and $\varepsilon>0$. So, integrability of (16) follows, providing at the same time that, for all τ, $\lambda>0$, we have $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)\in(0,\infty)$.

Finally, taking into account that

$$\begin{aligned} \varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) &= (2\pi\det\Sigma_u)^{-d/2} e^{-u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/2p_u^2} \\ &\sim (2\pi\det\Sigma)^{-d/2} e^{-u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/2(1-\lambda_u-\tau_u)} \\ &= (2\pi\det\Sigma)^{-d/2} e^{-u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}(1+\lambda_u+\tau_u+O(1/u^4))/2} \\ &\sim \varphi_\Sigma(u\widetilde{\boldsymbol{b}}) e^{-\frac{\lambda+\tau}{2}\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}} \quad \text{as } u\to\infty \end{aligned}$$

and substituting this into (15) we conclude the proof. □

Corollary 3.2. *For any λ positive and any $\boldsymbol{b}\in\mathbb{R}^d\setminus(-\infty,0]^d$ we have*

$$\lim_{\tau\to\infty}\frac{1}{\tau}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda) \ =:\ \mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda)\in[0,\infty). \tag{17}$$

Proof of Corollary 3.2: For any fixed τ_1, τ_2 and $\lambda>0$

$$\begin{aligned} & \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\le(\tau_1+\tau_2)/u^2,\ s\in[1-\lambda/u^2,1]\right\} \\ & \qquad\le\ \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\le\tau_1/u^2,\ s\in[1-\lambda/u^2,1]\right\} \\ & \qquad\quad +\ \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\le\tau_2/u^2,\ s\in[1-\lambda/u^2,1]\right\} \end{aligned}$$

by the stationarity of $\boldsymbol{Y}(t,s)$ with respect to t. Together with Lemma 3.1 this implies sub-additivity of $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)$ with respect to $\tau>0$. Hence by Fekete's lemma

$$\lim_{\tau\to\infty}\frac{1}{\tau}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda) \;=\; \inf_{\tau>0}\frac{1}{\tau}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda) \;\le\; \mathcal{H}_{\Sigma,\boldsymbol{b}}(1,\lambda)<\infty$$

establishing the claim. □

Corollary 3.3. *For all $\lambda>0$, we have*

$$\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b} \text{ for some } 0\le t\le T,\ s\in[1-\lambda/u^2,1]\right\} \;\le\; (T+o(1))u^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda)$$

as $u\to\infty$ uniformly for all T such that $\lim_{u\to\infty}Tu^2=\infty$.

Proof of Corollary 3.3: We follow the same idea as given in the proof of the upper bound in Theorem D.2 in [17]. Indeed, noting that $\boldsymbol{Y}(t,s)$ is stationary with respect to t, by Lemma 3.1, for each $\tau>0$

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b} \text{ for some } t\le T,\ s\in[1-\lambda/u^2,1]\right\}\le\\
&\le\; \lceil Tu^2/\tau\rceil\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b} \text{ for some } t\le \tau/u^2,\ s\in[1-\lambda/u^2,1]\right\}\\
&=\; \lceil Tu^2/\tau\rceil\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)u^{-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})(1+o(1)).
\end{aligned}\tag{18}$$

Hence, pushing $\tau\to\infty$, the proof is completed by Corollary 3.2. □

3.2. **Proof of Asymptotically Sharp Upper Bound (10).** The upper bound (10) follows from Corollary 3.2 and Lemma 3.4 below. For the proof of the aforementioned lemma we need to show the claim of Theorem 1.1. We tacitly assume that t, $t_k\ge 0$; for notation simplicity this is not mentioned everywhere.

Proof of Theorem 1.1: We consider first the case $m=1$. If $\boldsymbol{b}\le\boldsymbol{0}$, then the statement is immediate because then the right hand side is at least 1. Consider now the opposite case, so $\boldsymbol{0}\notin\boldsymbol{b}+V_1$ where $V_1=\{\boldsymbol{x}\ge\boldsymbol{0}\}$ is the positive orthant. Next, if $\theta:=\inf\{t>0:\boldsymbol{X}(t)\ge\boldsymbol{b}\}$, then the Markov property and the stationarity of the increments of $\boldsymbol{X}$ together with the continuity of the trajectory of $\boldsymbol{X}$ imply that

$$\mathbb{P}\left\{\boldsymbol{X}(T)\ge\boldsymbol{b}\right\} \;=\; \int_0^T\mathbb{P}\left\{\theta\in \mathrm{d}t\right\}\int_{\boldsymbol{b}+\partial V_1}\mathbb{P}\left\{\boldsymbol{X}(t)\in \mathrm{d}\boldsymbol{u}\right\}\mathbb{P}\left\{\boldsymbol{X}(T-t)\ge\boldsymbol{b}-\boldsymbol{u}\right\}.$$

Since $\boldsymbol{b}-\boldsymbol{u}\le\boldsymbol{0}$ on $\boldsymbol{b}+\partial V_1$, we have

$$\mathbb{P}\left\{\boldsymbol{X}(T-t)\ge\boldsymbol{b}-\boldsymbol{u}\right\} \;\ge\; \mathbb{P}\left\{\boldsymbol{X}(T-t)\ge\boldsymbol{0}\right\}.$$

Moreover, by the self-similarity of $\boldsymbol{X}$

$$\mathbb{P}\left\{\boldsymbol{X}(T-t)\ge\boldsymbol{0}\right\} \;=\; \mathbb{P}\left\{\boldsymbol{X}(1)\ge\boldsymbol{0}\right\}.\tag{19}$$

Consequently,

$$\mathbb{P}\left\{\boldsymbol{X}(T)\ge\boldsymbol{b}\right\} \;\ge\; \int_0^T\mathbb{P}\left\{\theta\in \mathrm{d}t\right\}\mathbb{P}\left\{\boldsymbol{X}(1)\ge\boldsymbol{0}\right\}$$

establishing the claim.

For simplicity, we only show next the case $m=2$. As above, a non-trivial case is where $\boldsymbol{0}\notin\boldsymbol{b}+V_1$. Fix a trajectory $\{\boldsymbol{x}_1(t_1),\ t_1\le T_1\}$ of $\boldsymbol{X}_1(t_1)$ and consider an event

$$\{\boldsymbol{X}_2(t_2)\ge\boldsymbol{b}-\boldsymbol{x}_1(t_1) \text{ for some } t_1\le T_1,\ t_2\le T_2\},$$

where randomness only comes from the $\boldsymbol{X}_2(t_2)$. Similar to how it is done in the last proof, we introduce a Markov stopping time

$$\theta_2 := \inf\{t_2 > 0 : \boldsymbol{X}_2(t_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1\}.$$

By the Markov property applied to $\boldsymbol{X}_2(t_2)$ and the stationarity of the increments of $\boldsymbol{X}_2$ we conclude that

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{X}_2(T_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1\right\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{X}_2(T_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1,\, \theta_2 \leq T_2\right\} \\
&\quad \geq \mathbb{P}\left\{\boldsymbol{X}_2(T_2) - \boldsymbol{X_2}(\theta_2) \geq \boldsymbol{0},\, \theta_2 \leq T_2\right\} \\
&\quad = \int_0^{T_2} \mathbb{P}\left\{\theta_2 \in \mathrm{d}t_2\right\} \mathbb{P}\left\{\boldsymbol{X_2}(T_2 - t_2) \geq \boldsymbol{0}\right\}.
\end{aligned}$$

As in (19)

$$\mathbb{P}\left\{\boldsymbol{X}_2(T_2 - t_2) \geq \boldsymbol{0}\right\} \;=\; \mathbb{P}\left\{\boldsymbol{X}_2(1) \geq \boldsymbol{0}\right\} \;=\; \mathbb{P}\left\{\boldsymbol{X}(1) \geq \boldsymbol{0}\right\},$$

hence we obtain

$$\mathbb{P}\left\{\boldsymbol{X}_2(T_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1\right\} \;\geq\; \int_0^{T_2} \mathbb{P}\left\{\theta_2 \in \mathrm{d}t\right\} \mathbb{P}\left\{\boldsymbol{X}(1) \geq \boldsymbol{0}\right\},$$

or, in other words

$$\mathbb{P}\left\{\boldsymbol{X}_2(t_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1,\; t_2 \leq T_2\right\} \;\leq\; \frac{\mathbb{P}\left\{\boldsymbol{X}_2(T_2) \geq \boldsymbol{b} - \boldsymbol{x}_1(t_1) \text{ for some } t_1 \leq T_1\right\}}{\mathbb{P}\left\{\boldsymbol{X}(1) \geq \boldsymbol{0}\right\}}.$$

Applying the same arguments to $\boldsymbol{X}_1(t_1)$ we complete the proof. □

Now we are ready to prove the remaining upper bound required to conclude the sharp upper bound (10).

Lemma 3.4. *There exists a constant $c < \infty$ such that, for all sufficiently large $\lambda > 0$*

$$\limsup_{u\to\infty} \sup_{T>1/u^2} \frac{u^{|I|-2}}{T\varphi_\Sigma(u\widetilde{\boldsymbol{b}})} \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \leq t \leq T,\; s \leq 1 - \lambda/u^2\right\} \;\leq\; ce^{-\frac{\lambda}{2}\boldsymbol{b}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I}.$$

Proof of Lemma 3.4: Slotting the interval $[0, T]$ onto Tu^2 small intervals of length $1/u^2$ each and making use of stationarity of $\boldsymbol{X}(t+s) - \boldsymbol{X}(t)$ with respect to t, we see that it suffices to prove the following result:

$$\limsup_{u\to\infty} \frac{u^{|I|}}{\varphi_\Sigma(u\widetilde{\boldsymbol{b}})} \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \leq 1/u^2,\; s \leq 1 - \lambda/u^2\right\} \;\leq\; c_1 e^{-\frac{\lambda}{2}\boldsymbol{b}_I^\top \Sigma_{II}^{-1}\boldsymbol{b}_I} \tag{20}$$

for some $c_1 < \infty$. We start with the inequality

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \leq 1/u^2,\; s \leq 1 - \lambda/u^2\right\} \\
&\quad \leq \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \leq 1/u^2,\; s \leq 1/u^2\right\} \\
&\quad + \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \leq 1/u^2,\; s \in [1/u^2, 1 - \lambda/u^2]\right\} \;=:\; P_1(u) + P_2(u).
\end{aligned} \tag{21}$$

Let us choose k such that $b_k > 0$. Then for all u positive we derive the following upper bound for $P_1(u)$

$$\begin{aligned} P_1(u) &\le \mathbb{P}\Big\{ \sup_{0\le t\le 1/u^2,\ 0\le s\le 1/u^2} Y_k(t,s) > ub_k \Big\} \\ &\le \mathbb{P}\Big\{ 2 \sup_{0\le t\le 2/u^2} |X_k(t)| > ub_k \Big\} \\ &\le 4\mathbb{P}\left\{ X_k(2/u^2) > ub_k/2 \right\}. \end{aligned}$$

Since $X_k(1)$ is normally distributed with mean zero, there exists an $\varepsilon > 0$ such that, for all sufficiently large u,

$$\mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le 1/u^2,\ 0 \le s \le 1/u^2 \right\} \le e^{-\varepsilon u^4}. \tag{22}$$

Hence we have

$$P_1(u) = o(\varphi_\Sigma(u\widetilde{\boldsymbol{b}})/u^{|I|}) \quad \text{as } u \to \infty.$$

In order to estimate $P_2(u)$, let us first notice that, for $t \in [0, 1/u^2]$ and $s \ge 1/u^2$

$$\boldsymbol{Y}(t,s) = (\boldsymbol{X}(1/u^2) - \boldsymbol{X}(t)) + (\boldsymbol{X}(t+s) - \boldsymbol{X}(1/u^2)),$$

where the two differences on the right hand side are independent random vectors. Therefore, we have

$$\begin{aligned} &\mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le 1/u^2,\ s \in [1/u^2, 1-\lambda/u^2] \right\} \\ &\qquad \le \mathbb{P}\left\{ \boldsymbol{X}'(t) + \boldsymbol{X}(s) > u\boldsymbol{b} \text{ for some } 0 \le t \le 1/u^2,\ 0 \le s \le 1-\lambda/u^2 \right\}, \end{aligned}$$

where $\boldsymbol{X}'$ and $\boldsymbol{X}$ are independent identically distributed processes. Next, by Theorem 1.1

$$\begin{aligned} &\mathbb{P}\left\{ \boldsymbol{X}'(t) + \boldsymbol{X}(s) > u\boldsymbol{b} \text{ for some } 0 \le t \le 1/u^2,\ 0 \le s \le 1-\lambda/u^2 \right\} \\ &\qquad\qquad \le c\mathbb{P}\left\{ \boldsymbol{X}(1-(\lambda-1)/u^2) > u\boldsymbol{b} \right\} \end{aligned}$$

for some $c < \infty$. Hence we obtain

$$\begin{aligned} &\mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le 1/u^2,\ s \in [1/u^2, 1-\lambda/u^2] \right\} \\ &\le c\mathbb{P}\left\{ \boldsymbol{X}(1-(\lambda-1)/u^2) > u\boldsymbol{b} \right\} = c\mathbb{P}\Big\{ \boldsymbol{X}(1) > \frac{u}{\sqrt{1-(\lambda-1)/u^2}} \boldsymbol{b} \Big\}. \end{aligned}$$

Since further for all $u > 0$

$$\left(\frac{u}{\sqrt{1-(\lambda-1)/u^2}} \right)^2 \ge u^2(1-(\lambda-1)/u^2) = u^2 + \lambda - 1,$$

then the above combined with (22) implies (20). □

Proof of the upper bound (10): Using that for each $\lambda < u^2$,

$$\begin{aligned} &\mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le T,\ 0 \le s \le 1 \right\} \\ &\le \mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le T,\ s \in [1-\lambda/u^2, 1] \right\} \\ &\quad + \mathbb{P}\left\{ \boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } 0 \le t \le T,\ 0 \le s \le 1-\lambda/u^2 \right\}, \end{aligned}$$

the proof follows straightforwardly by letting $\lambda \to \infty$ and combination of Corollary 3.3 with Lemma 3.4. The proof that $\mathcal{H}_{\Sigma,\boldsymbol{b}} \in (0,\infty)$ is postponed to Section 3.4. □

3.3. **Proof of the lower bound** (11)**:** We start with the following auxiliary result. For γ, τ, $\lambda > 0$, set

$$D_u(\gamma) \;:=\; [0,\tau/u^2]\times[1-\lambda/u^2,1]\times[\gamma/u^2,(\gamma+\tau)/u^2]\times[1-\lambda/u^2,1]$$

and event

$$B_u(\gamma) \;:=\; \{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t\in[\gamma/u^2,(\gamma+\tau)/u^2],\; s\in[1-\lambda/u^2,1]\}.$$

Let

$$\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda) \;:=\; e^{-\frac{5\lambda/2+3\tau}{4}\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}}\int_{\mathbb{R}^I} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I-\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J/2}$$
$$\times\mathbb{P}\bigg\{\bigg(\frac{\boldsymbol{X}_1(\tau,\tau{+}\lambda){+}\boldsymbol{X}_1(t,s)}{2}+\frac{\boldsymbol{X}_2(\tau,\tau{+}\lambda){+}\boldsymbol{X}_2(w,v)}{2}\bigg)_I > \boldsymbol{x}_I \text{ for some } t,w{\le}\tau,\; s,v{\le}\tau{+}\lambda\bigg\}\mathrm{d}\boldsymbol{x}_I,$$

where $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent copies of the process $\boldsymbol{X}$.

Lemma 3.5. *Under conditions of Lemma 3.1, for all τ and λ positive, there is a constant c such that*

$$\mathbb{P}\{B_u(0)\cap B_u(\gamma)\} \;\;\le\;\; cu^{-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})e^{-\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}\;\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda),$$

for all $\gamma\in[\tau+\lambda,u^2-\tau-\lambda]$ where

$$\limsup_{\tau\to\infty}\frac{1}{\tau^2}\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda) \;=:\; \widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\lambda)\in[0,\infty).$$

Proof of Lemma 3.5: If $\tau+\lambda<\gamma<u^2-\tau-\lambda$, then

$$\begin{array}{lll}(\tau/u^2,\gamma/u^2], & (\gamma/u^2,(\gamma+\tau)/u^2], & ((\gamma+\tau)/u^2,1-\lambda/u^2],\\ (1-\lambda/u^2,1+\tau/u^2], & (1+\tau/u^2,1+(\gamma-\lambda)/u^2], & (1+(\gamma-\lambda)/u^2,1+(\gamma+\tau)/u^2]\end{array}$$

are successive proper intervals, for all sufficiently large u. Then, for $(t,s,w,v)\in D_u(\gamma)$,

$$\begin{aligned}&\boldsymbol{X}(t+s)-\boldsymbol{X}(t)+\boldsymbol{X}(w+v)-\boldsymbol{X}(w)\\ &=\;[\boldsymbol{X}(\tau/u^2)-\boldsymbol{X}(t)]+[\boldsymbol{X}(\gamma/u^2)-\boldsymbol{X}(\tau/u^2)]+[\boldsymbol{X}((\gamma+\tau)/u^2)-\boldsymbol{X}(\gamma/u^2)]\\ &\quad+[\boldsymbol{X}(1-\lambda/u^2)-\boldsymbol{X}((\gamma+\tau)/u^2)]+[\boldsymbol{X}(t+s)-\boldsymbol{X}(1-\lambda/u^2)]\\ &\quad+[\boldsymbol{X}((\gamma+\tau)/u^2)-\boldsymbol{X}(w)]+[\boldsymbol{X}(1-\lambda/u^2)-\boldsymbol{X}((\gamma+\tau)/u^2)]+[\boldsymbol{X}(1+\tau/u^2)-\boldsymbol{X}(1-\lambda/u^2)]\\ &\quad+[\boldsymbol{X}(1+(\gamma-\lambda)/u^2)-\boldsymbol{X}(1+\tau/u^2)]+[\boldsymbol{X}(w+v)-\boldsymbol{X}(1+(\gamma-\lambda)/u^2)].\end{aligned}$$

Collecting together terms which do not depend on t, s, w and v and are independent of other terms we get the following representation for the right hand side

$$\begin{aligned}&2p_u(\gamma)A\boldsymbol{Z}+[\boldsymbol{X}(\tau/u^2)-\boldsymbol{X}(t)]+[\boldsymbol{X}((\gamma+\tau)/u^2)-\boldsymbol{X}(\gamma/u^2)]+[\boldsymbol{X}(t+s)-\boldsymbol{X}(1-\lambda/u^2)]\\ &+[\boldsymbol{X}((\gamma+\tau)/u^2)-\boldsymbol{X}(w)]+[\boldsymbol{X}(1+\tau/u^2)-\boldsymbol{X}(1-\lambda/u^2)]+[\boldsymbol{X}(w+v)-\boldsymbol{X}(1+(\gamma-\lambda)/u^2)],\end{aligned}$$

where $\boldsymbol{Z}$ is a standard normal random vector independent of all other random variables and

$$p_u^2(\gamma) \;=\; 1-(2\gamma+5\lambda+6\tau)/4u^2. \tag{23}$$

If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are two independent copies of the process $\boldsymbol{X}$ independent of $\boldsymbol{Z}$, then we have the following equivalent representation for the last random variable

$$\begin{aligned}&2p_u(\gamma)A\boldsymbol{Z}+[\boldsymbol{X}_1(\tau/u^2)-\boldsymbol{X}_1(t)]+[\boldsymbol{X}_2((\gamma+\tau)/u^2)-\boldsymbol{X}_2(\gamma/u^2)]+[\boldsymbol{X}_1(t+s)-\boldsymbol{X}_1(1-\lambda/u^2)]\\ &+[\boldsymbol{X}_2((\gamma{+}\tau)/u^2)-\boldsymbol{X}_2(w)]+[\boldsymbol{X}_1(1{+}\tau/u^2)-\boldsymbol{X}_1(1{-}\lambda/u^2)]+[\boldsymbol{X}_2(w{+}v)-\boldsymbol{X}_2(1{+}(\gamma{-}\lambda)/u^2)],\end{aligned}$$

which in its turn by the stationarity of increments implies the following equality in law

$$\begin{aligned}
&\boldsymbol{X}(t+s)-\boldsymbol{X}(t)+\boldsymbol{X}(w+v)-\boldsymbol{X}(w)\\
&\stackrel{d}{=}\ 2\boldsymbol{Z}_u+[\boldsymbol{X}_1(t+s-1+(\tau+\lambda)/u^2)-\boldsymbol{X}_1(t)]+[\boldsymbol{X}_1((2\tau+\lambda)/u^2)-\boldsymbol{X}_1(\tau/u^2)]\\
&\qquad+[\boldsymbol{X}_2(w+v-1+(\tau+\lambda)/u^2)-\boldsymbol{X}_2(w)]+[\boldsymbol{X}_2((\gamma+2\tau)/u^2)-\boldsymbol{X}_2((\gamma+\tau)/u^2)],
\end{aligned}$$

where $\boldsymbol{Z}_u := p_u(\gamma)A\boldsymbol{Z}$; let $\Sigma_u := p_u^2\Sigma$ be the covariance matrix of $\boldsymbol{Z}_u$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left\{B_u(0)\cap B_u(\gamma)\right\}\\
&\le\ \mathbb{P}\left\{[\boldsymbol{X}(t+s)-\boldsymbol{X}(t)]+[\boldsymbol{X}(w+v)-\boldsymbol{X}(w)]>2u\boldsymbol{b}\text{ for some }(t,s,w,v)\in D_u(\gamma)\right\}\\
&=\ \mathbb{P}\Big\{2\boldsymbol{Z}_u+[\boldsymbol{X}_1((2\tau+\lambda)/u^2)-\boldsymbol{X}_1(\tau/u^2)]+[\boldsymbol{X}_1(t+s)-\boldsymbol{X}_1(t)]\\
&+[\boldsymbol{X}_2(2\tau/u^2)-\boldsymbol{X}_2(\tau/u^2)]+[\boldsymbol{X}_2(w+v)-\boldsymbol{X}_2(w)]>2u\boldsymbol{b}\\
&\qquad\qquad\text{for some }t,w\le\tau/u^2\le s,v\le(\tau+\lambda)/u^2\Big\}\\
&=\ \mathbb{P}\left\{\boldsymbol{Z}_u+\frac{\boldsymbol{Y}_1(\tau/u^2,(\tau+\lambda)/u^2)+\boldsymbol{Y}_1(t,s)}{2}+\frac{\boldsymbol{Y}_2(\tau/u^2,\tau/u^2)+\boldsymbol{Y}_2(w,v)}{2}>u\boldsymbol{b}\right.\\
&\qquad\qquad\left.\text{for some }t,w\le\tau/u^2\le s,v\le(\tau+\lambda)/u^2\right\},
\end{aligned}$$

where $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are independent copies of $\boldsymbol{Y}$ independent of $\boldsymbol{Z}_u$. Let us now adapt calculations used in the proof of Lemma 3.1 to the evaluation of the probability on the right hand side. We have

$$\begin{aligned}
&\mathbb{P}\left\{B_u(0)\cap B_u(\gamma)\right\}\\
&\le\ \int_{\mathbb{R}^d}\varphi_{\Sigma_u}(-\boldsymbol{w})\mathbb{P}\left\{\frac{\boldsymbol{Y}_1(\tau/u^2,(\tau+\lambda)/u^2)+\boldsymbol{Y}_1(t,s)}{2}+\frac{\boldsymbol{Y}_2(\tau/u^2,\tau/u^2)+\boldsymbol{Y}_2(w,v)}{2}>u\boldsymbol{b}+\boldsymbol{w}\right.\\
&\qquad\qquad\left.\text{for some }t,w\le\tau/u^2\le s,v\le(\tau+\lambda)/u^2\right\}\mathrm{d}\boldsymbol{w}.
\end{aligned}$$

Let $\overline{\boldsymbol{u}}\in\mathbb{R}^d$ be a vector with coordinates $\overline{\boldsymbol{u}}_i=u$ for all $i\in I$ and $\overline{\boldsymbol{u}}_j=1$ else. Change of variables $\boldsymbol{w}=-u\widetilde{\boldsymbol{b}}+\boldsymbol{x}/\overline{\boldsymbol{u}}$, $\mathrm{d}\boldsymbol{w}=\mathrm{d}\boldsymbol{x}/u^{|I|}$, gives the following value of the last integral

$$\begin{aligned}
&u^{-|I|}\int_{\mathbb{R}^d}\varphi_{\Sigma_u}\Big(u\widetilde{\boldsymbol{b}}-\frac{\boldsymbol{x}}{\overline{\boldsymbol{u}}}\Big)\mathbb{P}\left\{\frac{\boldsymbol{Y}_1(\tau/u^2,(\tau+\lambda)/u^2)+\boldsymbol{Y}_1(t,s)}{2}+\frac{\boldsymbol{Y}_2(\tau/u^2,\tau/u^2)+\boldsymbol{Y}_2(w,v)}{2}\right.\\
&\qquad\qquad\left.>u(\boldsymbol{b}-\widetilde{\boldsymbol{b}})+\frac{\boldsymbol{x}}{\overline{\boldsymbol{u}}}\text{ for some }t,w\le\tau/u^2\le s,v\le(\tau+\lambda)/u^2\right\}\mathrm{d}\boldsymbol{x}\\
&=\ u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}})\int_{\mathbb{R}^d}e^{-\left((u\widetilde{\boldsymbol{b}}-\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(u\widetilde{\boldsymbol{b}}-\boldsymbol{x}/\overline{\boldsymbol{u}})-(u\widetilde{\boldsymbol{b}})^\top\Sigma^{-1}(u\widetilde{\boldsymbol{b}})\right)/2p_u^2}\\
&\qquad\times\mathbb{P}\left\{\widehat{\boldsymbol{Y}}(\lambda,\tau,s,t,v,w)>u^2(\boldsymbol{b}-\widetilde{\boldsymbol{b}})+\frac{u\boldsymbol{x}}{\overline{\boldsymbol{u}}}\text{ for some }t,w\le\tau\le s,v\le\tau+\lambda\right\}\mathrm{d}\boldsymbol{x},
\end{aligned}$$

where we set

$$\widehat{\boldsymbol{Y}}(\lambda,\tau,s,t,v,w)\ :=\ \frac{\boldsymbol{Y}_1(\tau,\tau+\lambda)+\boldsymbol{Y}_1(t,s)+\boldsymbol{Y}_2(\tau,\tau)+\boldsymbol{Y}_2(w,v)}{2}.$$

Then (13) implies that

$$\begin{aligned}
&\mathbb{P}\{B_u(0)\cap B_u(\gamma)\}\\
&\le u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}})\int_{\mathbb{R}^I} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2}\,\mathrm{d}\boldsymbol{x}_I\int_{\mathbb{R}^J} e^{-(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})/2p_u^2}\\
&\qquad\times\mathbb{P}\Big\{\widehat{\boldsymbol{Y}}(\lambda,\tau,s,t,v,w)_I > u^2(\boldsymbol{b}-\widetilde{\boldsymbol{b}})_I+\frac{u\boldsymbol{x}_I}{\overline{\boldsymbol{u}}_I}\text{ for some } t,w\le\tau\le s, v\le\tau+\lambda\Big\}\,\mathrm{d}\boldsymbol{x}_J\\
&= u^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}})\int_{\mathbb{R}^I} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I/p_u^2}\,\mathrm{d}\boldsymbol{x}_I\int_{\mathbb{R}^J} e^{-(\boldsymbol{x}/\overline{\boldsymbol{u}})^\top\Sigma^{-1}(\boldsymbol{x}/\overline{\boldsymbol{u}})/2p_u^2}\\
&\qquad\times\mathbb{P}\Big\{\widehat{\boldsymbol{Y}}(\lambda,\tau,s,t,v,w)_I > \boldsymbol{x}_I\text{ for some } t,w\le\tau\le s, v\le\tau+\lambda\Big\}\,\mathrm{d}\boldsymbol{x}_J,
\end{aligned}$$

because $(\boldsymbol{b}-\widetilde{\boldsymbol{b}})_i=0$ for all $i\in I$ (see Lemma A.1). Since $\lim_{u\to\infty}p_u=1$, the double integral above converges to

$$\begin{aligned}
&\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda)\\
&:= \int_{\mathbb{R}^d} e^{\boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I-\boldsymbol{x}_J^\top\Sigma_{JJ}^{-1}\boldsymbol{x}_J/2}\mathbb{P}\Big\{\widehat{\boldsymbol{Y}}(\lambda,\tau,s,t,v,w)_I > \boldsymbol{x}_I\text{ for some } t,w\le\tau\le s, v\le\tau+\lambda\Big\}\,\mathrm{d}\boldsymbol{x}
\end{aligned}$$

as $u\to\infty$. Therefore, for some $c<\infty$ which does not depend on γ

$$\mathbb{P}\{B_u(0)\cap B_u(\gamma)\} \le cu^{-|I|}\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}})\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda), \tag{24}$$

where we used the dominated convergence theorem, which may be justified in the same way as in the proof of Lemma 3.1. For p_u defined in (23) we get the following upper bound

$$\begin{aligned}
\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) &= \frac{1}{p_u(2\pi\det\Sigma)^{d/2}}\exp\Big\{-\frac{u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}}{2(1-(2\gamma+5\lambda+6\tau)/4u^2)}\Big\}\\
&\le \frac{1}{p_u(2\pi\det\Sigma)^{d/2}}\exp\Big\{-\frac{u^2\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}(1+(2\gamma+5\lambda+6\tau)/4u^2)}{2}\Big\}\\
&= \frac{1}{p_u}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\exp\Big\{-\frac{2\gamma+5\lambda+6\tau}{8}\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}\Big\}.
\end{aligned}$$

Thus, for $p_u>1/2$ we have

$$\varphi_{\Sigma_u}(u\widetilde{\boldsymbol{b}}) \le 2\varphi_\Sigma(u\widetilde{\boldsymbol{b}})e^{-\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4},$$

and substituting the above into (24) we conclude the proof. □

Proof of bound (11): Take $\gamma=\tau+\lambda$ and for simplicity assume that Tu^2/γ is a positive integer. By standard arguments we have

$$\begin{aligned}
\mathbb{P}\{\boldsymbol{Y}(t,s)>u\boldsymbol{b}\text{ for some } t\le T, s\le 1\} &\ge \mathbb{P}\Big\{\bigcup_{j=0}^{Tu^2/\gamma}B_u(j\gamma)\Big\}\\
&\ge \sum_{0\le j\le Tu^2/\gamma}\mathbb{P}\{B_u(j\gamma)\}-\sum_{0\le j<i\le Tu^2/\gamma}\mathbb{P}\{B_u(j\gamma)\cap B_u(i\gamma)\}.
\end{aligned}\tag{25}$$

For any fixed τ and λ, by Lemma 3.1,

$$\sum_{0\le j\le Tu^2/\gamma}\mathbb{P}\{B_u(j\gamma)\} \sim \frac{Tu^2}{\tau+\lambda}u^{-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)\quad\text{as } u\to\infty. \tag{26}$$

Then, by Lemma 3.5

$$\sum_{0\le j<i\le Tu^2/\gamma} \mathbb{P}\left\{B_u(j\gamma)\cap B_u(i\gamma)\right\} \le cu^{-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})\ \widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda) \sum_{0\le j<i\le Tu^2/\gamma} e^{-(i-j)\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}. \tag{27}$$

The sum on the right hand side is not greater than

$$\frac{Tu^2}{\gamma}\sum_{i=1}^{\infty} e^{-i\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4} \le \frac{Tu^2}{\tau+\lambda}\frac{e^{-\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}}{1-e^{-\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}} = c_1\frac{Tu^2}{\tau+\lambda}e^{-\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}.$$

Substituting the last upper bound into (27) and taking into account (26), we get from (25) that, for all fixed τ and λ

$$\liminf_{u\to\infty}\frac{u^{|I|-2}}{\varphi_\Sigma(u\widetilde{\boldsymbol{b}})}\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b}\text{ for some } t\le T,\ s\in[1-\lambda/u^2,1]\right\}$$
$$\ge \frac{T}{\tau+\lambda}\Big(\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)-c_1\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau,\lambda)e^{-\gamma\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}\Big). \tag{28}$$

Letting now $\tau\to\infty$ (and hence $\gamma\to\infty$) we get, for all fixed $\lambda>0$

$$\liminf_{u\to\infty}\frac{u^{|I|-2}}{\varphi_\Sigma(u\widetilde{\boldsymbol{b}})}\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b}\text{ for some } t\le T, s\le 1\right\} \ge T\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda),$$

which establishes the lower bound (11). □

3.4. **Positivity and finiteness of $\mathcal{H}_{\Sigma,\boldsymbol{b}}$.** We conclude the proof of Theorem 2.1 with the lemma that confirms that $\mathcal{H}_{\Sigma,\boldsymbol{b}}$ is positive and finite.

Lemma 3.6. *For any $\boldsymbol{b}\in\mathbb{R}^d\setminus(-\infty,0]^d$, we have $\mathcal{H}_{\Sigma,\boldsymbol{b}}\in(0,\infty)$.*

Proof of Lemma 3.6:

i) Proof that $\mathcal{H}_{\Sigma,\boldsymbol{b}}>0$: We begin with an observation that, by Lemma 3.1, for each $\tau>0$, $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)$ is an increasing function of λ. Thus it suffices to check that $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\lambda)>0$ for some $\lambda>0$.

Let τ, $\tau'>0$. From (18) we have

$$\limsup_{u\to\infty}\frac{\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b}\text{ for some } t\le T,\ s\in[1-\lambda/u^2,1]\right\}}{Tu^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})} \le \frac{\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)}{\tau},$$

while from (28)

$$\liminf_{u\to\infty}\frac{\mathbb{P}\left\{\boldsymbol{Y}(t,s)>u\boldsymbol{b}\text{ for some } t\le T,\ s\in[1-\lambda/u^2,1]\right\}}{Tu^{2-|I|}\varphi_\Sigma(u\widetilde{\boldsymbol{b}})}$$
$$\ge \frac{\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau',\lambda)-c_1\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau',\lambda)e^{-(\tau'+\lambda)\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}}{\tau'+\lambda}.$$

Hence

$$\frac{\tau'+\lambda}{\tau}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda) \ge \mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau',\lambda)-c_1\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau',\lambda)e^{-(\tau'+\lambda)\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4}. \tag{29}$$

Now it suffices to note that, by Lemma 3.5 we have

$$\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau',\lambda) \le c_2\tau'^2,$$

so

$$\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau',\lambda)-c_1\widehat{\mathcal{H}_{\Sigma,\boldsymbol{b}}}(\tau',\lambda)e^{-(\tau'+\lambda)\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4} \ge \mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau',\lambda)-c_3\tau'^2e^{-(\tau'+\lambda)\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4} > 0, \tag{30}$$

where the positivity follows from the fact that $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau',\lambda)$ is increasing as a function of τ' and $\tau'^2 e^{-(\tau'+\lambda)\widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}}/4} \to 0$ as $\tau' \to \infty$.

Thus, combination of (29) with (30) for appropriately large τ' and $\tau \to \infty$ confirms that $\mathcal{H}_{\Sigma,\boldsymbol{b}} > 0$.

ii) Proof that $\mathcal{H}_{\Sigma,\boldsymbol{b}} < \infty$: Using that, for each λ, $\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)$ is subadditive as a function of τ, and hence $\tau^{-1}\mathcal{H}_{\Sigma,\boldsymbol{b}}(\tau,\lambda)$ is nonincreasing as a function of τ, it suffices to prove that for $\tau = 1$

$$\lim_{\lambda\to\infty} \mathcal{H}_{\Sigma,\boldsymbol{b}}(1,\lambda) \quad < \quad \infty. \tag{31}$$

Indeed,

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le 1/u^2,\ s \in [1-(\lambda_1+\lambda_2)/u^2, 1]\right\} \\
&\le \quad \mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le 1/u^2,\ s \in [1-\lambda_1/u^2, 1]\right\} \\
&\quad +\mathbb{P}\left\{\boldsymbol{Y}(t,s) > u\boldsymbol{b} \text{ for some } t \le 1/u^2,\ s \le 1-\lambda_1/u^2\right\},
\end{aligned}$$

for all λ_1 and $\lambda_2 > 0$. Then it is straightforward by Lemma 3.1 and (20) that for each $\lambda_2 > 0$

$$\mathcal{H}_{\Sigma,\boldsymbol{b}}(1,\lambda_1+\lambda_2) \ \le\ \mathcal{H}_{\Sigma,\boldsymbol{b}}(1,\lambda_1) + c_1 e^{-\lambda_1(\Sigma_{II}^{-1}\boldsymbol{b}_I,\boldsymbol{b}_I)/2},$$

which confirms the existence of the limit in (31) and its finiteness. □

Appendix A. Quadratic programming problems

The next result is known and formulated for instance in [7].

Lemma A.1. *Let Σ be a positive definite matrix of size $d\times d$ with inverse Σ^{-1}. If $\boldsymbol{b} \in \mathbb{R}^d \setminus (-\infty,0]^d$, then the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$ formulated in (3) has a unique solution $\widetilde{\boldsymbol{b}}$ and there exists a unique non-empty index set $I \subseteq \{1,\dots,d\}$ with $m \le d$ elements such that*

$$\widetilde{\boldsymbol{b}}_I = \boldsymbol{b}_I,\ \text{and if } J := \{1,\dots,d\}\setminus I \neq \emptyset,\ \text{then } \widetilde{\boldsymbol{b}}_J \quad = \quad \Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{b}_I \ge \boldsymbol{b}_J, \quad \Sigma_{II}^{-1}\boldsymbol{b}_I > \boldsymbol{0}_I, \tag{32}$$

$$\min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x} = \widetilde{\boldsymbol{b}}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}} \quad = \quad \boldsymbol{b}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I > 0. \tag{33}$$

Furthermore, for any $\boldsymbol{x} \in \mathbb{R}^d$ we have

$$\boldsymbol{x}^\top\Sigma^{-1}\widetilde{\boldsymbol{b}} = \boldsymbol{x}_I^\top\Sigma_{II}^{-1}\widetilde{\boldsymbol{b}}_I = \boldsymbol{x}_I^\top\Sigma_{II}^{-1}\boldsymbol{b}_I \tag{34}$$

and if $\boldsymbol{b} = c\boldsymbol{1}, c \in (0,\infty)$, then $2 \le |I| \le k$ and J is empty if $\Sigma^{-1}\boldsymbol{b} > \boldsymbol{0}$.

Acknowledgments

The authors would like to thank two anonymous referees for their valuable comments and suggestions to improve the quality of the paper. Support from SNSF Grant no. 200021-175752/1 is kindly acknowledged.

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