CLIFFORD ALGEBRA ANALOGUE OF CARTAN'S THEOREM FOR SYMMETRIC PAIRS

KIERAN CALVERT, KARMEN GRIZELJ, ANDREY KRUTOV, AND PAVLE PANDŽIĆ

ABSTRACT. We extend Kostant's results about $\mathfrak{g}$-invariants in the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ to the relative case of $\mathfrak{k}$-invariants in the Clifford algebra $\mathrm{Cl}(\mathfrak{p})$, where $(\mathfrak{g},\mathfrak{k})$ is a classical symmetric pair and $\mathfrak{p}$ is the (-1)-eigenspace of the corresponding involution. In this setup we prove the Cartan theorem for Clifford algebras, a relative transgression theorem, the Harish–Chandra isomorphism for $\mathrm{Cl}(\mathfrak{p})$, and a relative version of Kostant's Clifford algebra conjecture.

1. Introduction

Let G be a compact simple connected Lie group and K a closed subgroup (hence a Lie subgroup). We do not assume K is connected, but we will be assuming that K is a symmetric subgroup, i.e., there is an involution Θ of G such that

$$G_0^\Theta \subseteq K \subseteq G^\Theta,$$

where G^Θ denotes the subgroup of G consisting of the points fixed under Θ, and G_0^Θ denotes the connected component of G^Θ. Typical examples of compact symmetric spaces G/K are the classical ones, such as $SU(p+q)/S(U(p)\times U(q))$, $SO(p+q)/S(O(p)\times O(q))$, $Sp(p+q)/Sp(p)\times Sp(q)$. Among these, $K=S(O(p)\times O(q))$ is not connected. Note that in all these examples $K=G^\Theta$ for an appropriate Θ

By well known classical results of É. Cartan and de Rham, the de Rham cohomology (with complex coefficients) $H(G/K)$ of G/K as above can be described as follows. Let B be a nondegenerate invariant symmetric bilinear form on the complexified Lie algebra $\mathfrak{g}$ of G; such a form exists since G is reductive, so one can take the Killing form on the semisimple part of $\mathfrak{g}$ and extend by any nondegenerate form on the center of $\mathfrak{g}$. Let $\mathfrak{k}\subseteq\mathfrak{g}$ be the complexified Lie algebra of K and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ with respect to B, so that

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p},$$

and this decomposition is compatible with the adjoint K-action. The space $\mathfrak{p}$ can also be described as the (-1)-eigenspace of $\theta=\mathrm{d}\,\Theta$. Denote by $(\bigwedge\mathfrak{p})^K$ the algebra

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of K-invariants in $\bigwedge\mathfrak{p}$. Then

$$H(G/K) = (\bigwedge\mathfrak{p})^K$$

as graded algebras.

As mentioned above, this result is old and very well known, but it is not easy to find a proof in the literature. One place where one can find it is [CNP, §4.1].

The structure of the algebra $(\bigwedge\mathfrak{p})^{\mathfrak{k}}$ is well understood through the work of Borel, H. Cartan, Hopf, Koszul, Samelson and others, see [Car1, Car2, Bor, And, Ras]. This is achieved using a series of quasi-isomorphisms and it is quite involved and not completely explicit; moreover, there is no treatment of $(\bigwedge\mathfrak{p})^K$ for disconnected K. See [Oni, 12.3 Theorem 1] and [GHV, X.2 Theorem III].

On the other hand, one can replace the algebra $\bigwedge\mathfrak{p}$ by its filtered deformation (or quantisation), the Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ with respect to B. This is an associative algebra with unit, generated by $\mathfrak{p}$, with relations

$$XY + YX = 2B(X,Y), \qquad X,Y \in \mathfrak{p}.$$

Clearly, $\mathrm{Cl}(\mathfrak{p})$ is a quotient of the tensor algebra $T(\mathfrak{p})$ by a nonhomogeneous ideal, so it inherits a filtration, and the corresponding graded algebra is $\bigwedge\mathfrak{p}$.

One might think that the filtered algebras $\mathrm{Cl}(\mathfrak{p})$ and $\mathrm{Cl}(\mathfrak{p})^K$ are more complicated than their associated graded algebras $\bigwedge\mathfrak{p}$ and $(\bigwedge\mathfrak{p})^K$, but in many aspects the filtered versions are in fact simpler. To illustrate this, let us temporarily assume that $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank, and let $\mathfrak{t}$ be a common Cartan subalgebra. Then $\dim\mathfrak{p}$ is even, and the Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ has only one simple module, the spin module S; moreover, any $\mathrm{Cl}(\mathfrak{p})$-module is a direct sum of copies of S, and $\mathrm{Cl}(\mathfrak{p}) \cong \operatorname{End} S$. To construct S, write

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are maximal isotropic subspaces of $\mathfrak{p}$ in duality under B. Then one can take $S = \bigwedge\mathfrak{p}^+$, with elements of $\mathfrak{p}^+$ acting by wedging, and elements of $\mathfrak{p}^-$ by contracting. The spin module S carries an action of the spin double cover $\widetilde{K}$ of K; the corresponding action of $\mathfrak{k}$ is described by the Lie algebra map $\alpha_{\mathfrak{p}} : \mathfrak{k} \to \mathrm{Cl}(\mathfrak{p})$,

$$\alpha_{\mathfrak{p}}(X) = \frac{1}{4}\sum_i [X, b_i] d_i, \qquad X \in \mathfrak{k},$$

where b_i is any basis of $\mathfrak{p}$ and d_i is the dual basis with respect to B. As a $\mathfrak{k}$-module, S is multiplicity free and decomposes as

$$S = \bigoplus_{w \in W^1} E_{w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}}.$$

Here $\rho_{\mathfrak{g}}$ and $\rho_{\mathfrak{k}}$ are the half sums of compatible positive roots of $(\mathfrak{g},\mathfrak{t})$ respectively $(\mathfrak{k},\mathfrak{t})$. Furthermore,

$$W^1 = \{w \in W_{\mathfrak{g}} \,|\, w\rho_{\mathfrak{g}} \text{ is } \mathfrak{k}\text{-dominant}\},$$

where $W_{\mathfrak{g}}$ denotes the Weyl group of $(\mathfrak{g},\mathfrak{t})$. Finally, $E_{w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}}$ denotes the irreducible finite-dimensional $\mathfrak{k}$-module with highest weight $w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$.

Now $\mathrm{Cl}(\mathfrak{p}) \cong \operatorname{End} S$ implies

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} = \operatorname{End}_{\mathfrak{k}} S,$$

and by Schur's Lemma this is the algebra of projections $\mathrm{pr}_w : S \to E_{w\rho_{\mathfrak{g}}-\rho_{\mathfrak{k}}}$. This is a very simple commutative algebra, isomorphic to $\mathbb{C}^{|W^1|}$ with coordinate-wise multiplication. (The isomorphism is given by identifying pr_w with $(0,\dots,0,1,0,\dots,0)$, with 1 in the place corresponding to w.) This was explained to the fourth-named author by Kostant [Kos4]; see [CGKP] and [CNP] for more details. We can now extend our map $\alpha_{\mathfrak{p}}$ to the universal enveloping algebra $U(\mathfrak{k})$ of $\mathfrak{k}$, restrict it to the center $Z(\mathfrak{k})$ of $U(\mathfrak{k})$, and identify $Z(\mathfrak{k})$ with $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ by the Harish-Chandra isomorphism (here $W_{\mathfrak{k}}$ is the Weyl group of $(\mathfrak{k},\mathfrak{t})$). In this way we get a map

$$\alpha_{\mathfrak{p}} : \mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}} \to \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}, \qquad \alpha_{\mathfrak{p}}(P) = \sum_{w\in W^1} P(w\rho)\,\mathrm{pr}_w,$$

which is easily seen to be onto (in the equal rank case). Moreover, the kernel of $\alpha_{\mathfrak{p}}$ is generated by the $W_{\mathfrak{g}}$-invariants in $\mathbb{C}[\mathfrak{t}^*]$ that evaluate to 0 at ρ, and thus one gets to understand $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ by generators and relations. One can then pass to associated graded algebras and obtain results about the structure of $(\bigwedge\mathfrak{p})^{\mathfrak{k}}$. This approach is much more direct and explicit than the classical approach described earlier, and has an additional advantage of simultaneously understanding the filtered deformation $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ of $(\bigwedge\mathfrak{p})^{\mathfrak{k}}$. Finally, it is not difficult to extend these considerations to the algebras $\mathrm{Cl}(\mathfrak{p})^K$ and $(\bigwedge\mathfrak{p})^K$ for disconnected K; see [CNP] and [CGKP].

If $\mathfrak{g}$ and $\mathfrak{k}$ do not have equal rank, $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ is typically much bigger than the algebra $\mathrm{Pr}(S)$ of projections of the spin module; moreover, it is not abelian any more. In fact, it is the tensor product of the algebra of projections and a Clifford algebra isomorphic to $\mathrm{Cl}(\mathfrak{a})$, where $\mathfrak{a}$ is the split part of the fundamental Cartan subalgebra $\mathfrak{h} = \mathfrak{t}\oplus\mathfrak{a}$ of $\mathfrak{g}$. The situation is thus analogous to the case of $(\bigwedge\mathfrak{p})^{\mathfrak{k}}$ studied by Borel, H. Cartan, Hopf, Koszul and Samelson. The Clifford algebra setting was first studied by Kostant [Kos2] who proved that in the "absolute" case when $\mathfrak{k}$ and $\mathfrak{p}$ are both isomorphic to a complex Lie algebra $\mathfrak{g}$ (and what we denoted by $\mathfrak{g}$ earlier is now $\mathfrak{g}\oplus\mathfrak{g}$), $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is isomorphic to the Clifford algebra $\mathrm{Cl}(\mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and this copy of $\mathrm{Cl}(\mathfrak{h})$ is realized inside $\mathrm{Cl}(\mathfrak{g})$ as a Clifford algebra over the primitive elements $P_\wedge(\mathfrak{g})$ which correspond to $\mathfrak{h}\subset\mathrm{Cl}(\mathfrak{h})$, but have various high degrees. This is analogous to the classical Hopf-Koszul-Samelson theorem about $(\bigwedge\mathfrak{g})^{\mathfrak{g}}$.

The purpose of this article is to prove results analogous to Kostant's in the "relative" case. We show that for a symmetric pair (G,K) as above, the algebra $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ is isomorphic to a tensor product of the algebra of projections of the spin module with a Clifford algebra isomorphic to $\mathrm{Cl}(\mathfrak{a})$. The subalgebra $\mathrm{Cl}(\mathfrak{a})$ is realized inside $\mathrm{Cl}(\mathfrak{p})$ as the Clifford algebra over the "primitives" $P_\wedge(\mathfrak{p})$ corresponding to $\mathfrak{a}$ but lying in various high degrees. Explicitly $P_\wedge(\mathfrak{p})$ is the projection of $P_\wedge(\mathfrak{g})$ to $(\bigwedge\mathfrak{p})^{\mathfrak{k}}$. We denote by $P_{\mathrm{Cl}}(\mathfrak{p})$ the image of $P_\wedge(\mathfrak{p})$ under the Chevalley map (skew-symmetrization) $q:\bigwedge\mathfrak{p}\to\mathrm{Cl}(\mathfrak{p})$. We define a form $\tilde{B}$ on the vector space $(\bigwedge\mathfrak{p})^{\mathfrak{k}}\cong\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ by setting $\tilde{B}(a,b)$ to be the 0th component of $\iota_a b$, where ι_a is the contraction of $\bigwedge\mathfrak{p}$ by a. We can now state our main theorem:

Theorem 1.1. *Let (G,K) be a compact symmetric pair such that G is simple and connected. Assume that $(\mathfrak{g},\mathfrak{k})$ is different from $(\mathfrak{e}(6),\mathfrak{sp}(8))$.*

(a) With the above notation, the inclusion $P_{\mathrm{Cl}}(\mathfrak{p})\hookrightarrow\mathrm{Cl}(\mathfrak{p})^K$ extends to a filtered algebra homormorpism

$$\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}),\tilde{B})\to\mathrm{Cl}(\mathfrak{p})^K,$$

which is an isomorphism in the primary cases, i.e., when the spin module S contains only one $\mathfrak{k}$-type. For the "almost primary cases" $(G,K)=(SU(2n),SO(2n))$, the statement remains true if K is replaced by $O(2n)$.

(b) There is an algebra isomorphism

$$\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}),\tilde{B})\otimes\Pr(S)\cong\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}.$$

(c) There is a filtered algebra isomorphism

$$\Pr(S)\cong\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}/I_\rho,$$

where I_ρ is the ideal of $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ generated by $a\in\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g}}}$ such that $a(\rho)=0$.

(d) Passing to the associated graded algebras we recover the well known Cartan (or Cartan-Borel) Theorem for symmetric spaces

$$\bigwedge P_\wedge(\mathfrak{p})\otimes\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}/I_+\cong(\bigwedge\mathfrak{p})^{\mathfrak{k}},$$

where $I_+=\mathrm{gr}\,I_\rho$ is the ideal of $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{k}}}$ generated by elements of $\mathbb{C}[\mathfrak{t}^*]^{W_{\mathfrak{g}}}$ with zero constant term.

For the equal rank case, Theorem 1.1(a) is trivial and (b) was proved by Kostant (see above). For the rest, see [CGKP, Section 2] and [CNP, Section 3.4]. For $(\mathfrak{g},\mathfrak{k})=(\mathfrak{so}(2p+2q+2,\mathbb{C}),\mathfrak{so}(2p+1,\mathbb{C})\oplus\mathfrak{so}(2q+1,\mathbb{C}))$ Theorem 1.1 is proved in [CNP, Section 3.5]; this case is called almost equal rank because $\dim\mathfrak{a}=1$. In this paper we prove this result in the remaining cases, the primary and almost primary cases.

One of Kostant's main tools in the absolute case was the Transgression Theorem, which allows one to construct primitive invariants in $\bigwedge\mathfrak{g}$ starting from the better known primitive invariants in the symmetric algebra $S(\mathfrak{g})$. Analogues of this theorem play also an important role in our approach. Kostant also used the Hodge decomposition and the corresponding duality for Lie algebra (co)homology. This is not available in the relative case, so we developed new tools based on transgression and Harish–Chandra maps. This also gives an alternative way to prove Kostant's results.

We revisit the transgression construction and show that it factorises via the noncommutative Weil algebra

$$\widetilde{W}(\mathfrak{g})=T(\mathfrak{g}[-2]\oplus\mathfrak{g}[-1])$$

defined in [AM3] (see Section 3.3). This allows us to show that the image of the transgression map consists of "primitives" a la [Ras] and obtain an alternative proof of Kostant's results which is more suitable for our generalisations.

In a similar manner we define transgression maps for the relative Weil algebras

$$W(\mathfrak{g},K)=W(\mathfrak{g})_{K-\mathrm{basic}}=(S(\mathfrak{g})\otimes\bigwedge\mathfrak{p})^K$$

and prove the relative version of the transgression theorem, Theorem 5.8. It says that we can produce primitive K-invariants in $\bigwedge\mathfrak{p}$ starting from those primitive $\mathfrak{g}$-invariants in $S(\mathfrak{g})$ that project trivially to $S(\mathfrak{k})$.

We also study other related absolute and relative Weil algebras, as well as corresponding transgression maps. For example, we consider the quantum Weil algebra

$$\mathcal{W}(\mathfrak{g})=U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{g})$$

and its relative version

$$\mathcal{W}(\mathfrak{g},K)=(U(\mathfrak{g})\otimes\mathrm{Cl}(\mathfrak{p}))^K.$$

To relate the results about transgression and Theorem 1.1, we use certain Harish-Chandra type projections of the Clifford algebra. In the absolute case, a Harish-Chandra map hc from $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ to $\mathrm{Cl}(\mathfrak{h})$ was studied by Kostant, Bazlov and others; see [Mei]. In the relative cases, we define analogues of this map attached to elements of W^1. We show that each $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$ is a surjective algebra map, and that hc_1 is an isomorphism in the primary cases; see Propositions 6.7 and 7.2.

In the absolute cases, it was conjectured by Kostant and proved by Bazlov [Baz2], Joseph [Jos] and Alekseev-Moreau [AM4], that hc is an isomorphism of filtered algebras, if we introduce a filtration of $\mathfrak{h}$ using the action of a principal $\mathfrak{sl}(2)$ subalgebra of the dual Lie algebra $\mathfrak{g}^\vee$. We show that an analogous result holds in primary and almost primary cases, see Theorem 8.2.

Proposition 6.7, Proposition 7.2 and Theorem 8.2 can be summarized as follows:

Theorem 1.2. *Let (G, K) be a symmetric pair as in Theorem 1.1. Let $\mathfrak{a}$ be filtered using the action of the principal $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{g}^\vee$. Then for each $w \in W^1$, $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$ is a surjective filtered algebra homomorphism. In the primary case, hc_1 is an isomorphism.*

The paper is organized as follows. After reviewing some preliminaries in Section 2, we discuss various Weil algebras and transgression maps in Section 3 and 4. Section 5 is devoted to the proof of Relative Transgression Theorem and in Section 6 we study the Harish-Chandra maps. In Section 7 we prove the main theorem, Theorem 1.1. Finally, in Section 8 we discuss the relative version of Kostant's Clifford algebra conjecture.

2. Preliminaries

In this section we survey some definitions and facts about Clifford algebras and symmetric pairs.

2.1. **Contractions and bilinear forms.** Let V be a vector space with a nondegenerated symmetric bilinear form B.

The transpose anti-automorphism $\bullet^T$ of $\bigwedge V$ is defined by sending $a = v_1 \wedge \ldots \wedge v_k \in \bigwedge V$ to $a^T = v_k \wedge \ldots \wedge v_1$. Since $\bigwedge V$ is super-commutative, $\bullet^T$ is multiplication by $(-1)^k$ on each graded component $\bigwedge^k V$.

Definition 2.1. For $v \in V$, we define a derivation $\iota_v : \bigwedge V \to \bigwedge V$ by

$$\iota_v(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^i B(v, v_i) v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_k.$$

The map ι extends to an algebra homomorphism $\bigwedge V \to \mathrm{End}(\bigwedge V)$ and is called contraction.

Definition 2.2. For $a \in \bigwedge V$ let $(a)_{[0]}$ be the 0^{th} degree part of a. We extend the bilinear form B to $\bigwedge V$ in two ways;

$$B(a, b) = (\iota_{a^T} b)_{[0]}, \quad \widetilde{B}(a, b) = (\iota_a b)_{[0]}.$$

2.2. **Clifford algebras.** In this section we recall definition and main properties of Clifford algebras. Our main reference is [Mei, Chapter 2]; see also [HP2, §2].

Definition 2.3. For a vector space V with symmetric nondegenerate inner product B, define the Clifford algebra

$$\mathrm{Cl}(V,B) = T(V)/\langle u \otimes v + v \otimes u - 2B(u,v)\rangle.$$

We often drop the bilinear form in notation and write $\mathrm{Cl}(V,B)$ as $\mathrm{Cl}(V)$.

Depending on the dimension of V, $\mathrm{Cl}(V)$ is either an endomorpism algebra or two copies of an endomorphism algebra of spin modules S and S_1, S_2 respectively.

$$\mathrm{Cl}(V) \cong \begin{cases} \mathrm{End}(S) & \text{if } \dim V \text{ is even,} \\ \mathrm{End}(S_1) \oplus \mathrm{End}(S_2) & \text{if } \dim V \text{ is odd.} \end{cases}$$

The associated graded algebra of the Clifford algebra $\mathrm{Cl}(V)$ is the exterior algebra $\bigwedge V$. These are isomorphic as vector spaces. The isomorphisms are given by the symbol map $\mathrm{symb} : \mathrm{Cl}(V) \to \bigwedge V$ and the quantization map $q : \bigwedge V \to \mathrm{Cl}(V)$; see [Mei, Section 2.2.5].

For $v \in V$ we define an odd derivation $\iota_v \colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ by

$$\iota_v(v_1 \dots v_k) = \sum_{i=1}^{k-1} B(v, v_i) v_1 \dots v_{i-1} v_{i+1} \dots v_k,$$

The quantisation map intertwines contraction operators in $\bigwedge V$ and $\mathrm{Cl}(V)$: for all $\omega \in \bigwedge V$, $v \in V$ we have that $\iota_v(q(\omega)) = q(\iota_v \omega)$.

The filtration on $\mathrm{Cl}(V)$ is compatible with the $\mathbb{Z}$-grading of $\bigwedge V$ in the following manner

$$\mathrm{Cl}^{(k)}(V) = q\left(\bigoplus_{m=0}^{k} \bigwedge^m V\right) \qquad \text{for } k = 0, \dots, \dim(V). \tag{2.4}$$

The degree 2 space $q(\bigwedge^2 V) \subseteq \mathrm{Cl}^{(2)}(V)$ is closed under the Lie bracket $[X,Y] = XY - YX$. The Lie algebra $q(\bigwedge^2 V)$ is isomorphic to $\mathfrak{o}(V)$.

Let L be a compact Lie group and $\mathfrak{l}$ be the corresponding Lie algebra (in examples we need, L will be G or K). Now assume that V is a $\mathfrak{l}$-module which admits a non-degenerate invariant symmetric bilinear form B. Since B is symmetric and invariant, the action of $\mathfrak{l}$ on V defines a Lie algebra homomorphism $\mathfrak{l} \to \mathfrak{o}(V)$. Hence we get a $\mathfrak{l}$-equivariant linear map

$$\lambda_V \colon \mathfrak{l} \longrightarrow \mathfrak{o}(V) \xrightarrow{\ \simeq\ } \textstyle\bigwedge^2 V.$$

Composing it with the embedding of the Lie algebra $\mathfrak{o}(V) \cong \bigwedge^2 V$ into the Clifford algebra $\mathrm{Cl}(V)$, we get a Lie algebra map

$$\alpha_V \colon \mathfrak{l} \longrightarrow \mathfrak{o}(V) \xrightarrow{\ \simeq\ } \textstyle\bigwedge^2 V \xrightarrow{\ q\ } \mathrm{Cl}^{(2)}(V). \tag{2.5}$$

Explicitly, if e_i is a basis of V with dual basis f_i with respect to B, then

$$\lambda_V(X) = \frac{1}{4} \sum_i^{(V)} [X, e_i] \wedge f_i, \qquad \alpha_V(X) = \frac{1}{4} \sum_i^{(V)} [X, e_i] f_i, \quad X \in \mathfrak{l},$$

where $\sum_i^{(V)}$ denotes the summation over a basis of V indexed by i. By the universal properties, λ_V and α_V extend to algebra maps

$$\lambda_V \colon S(\mathfrak{l}) \to \textstyle\bigwedge V, \qquad \alpha_V \colon U(\mathfrak{l}) \to \mathrm{Cl}(V),$$

(in particular, α_V defines a quantum moment map in sense of [Lu]). Therefore, the spin module S (or S_1 and S_2) of $\mathrm{Cl}(V)$ can be viewed as an $\mathfrak{l}$-module. We define $\alpha_Z = \alpha|_{Z(U(\mathfrak{l}))}$ to be the restriction of α_V to the centre of $U(\mathfrak{l})$.

In a similar manner one defines the corresponding $\tilde{L}$-module structure on S (or S_1, S_2), where $\tilde{L}$ is a double cover of L which makes the following diagram commutative

$$\begin{array}{ccc} \tilde{L} & \longrightarrow & \mathrm{Spin}(V) \\ \downarrow & & \downarrow \\ L & \longrightarrow & SO(V,B) \end{array}$$

See [HP2, §3.2.1] for details.

2.3. **Symmetric pairs of Lie algebras.** Let (G,K) be a symmetric pair of compact Lie groups as in the introduction, with Θ being the corresponding involution of G. Then the complexified Lie algebra $\mathfrak{g}$ of G has an involution $\theta = \mathrm{d}\,\Theta$ such that $\mathfrak{g}^\theta = \mathfrak{k}$ is the complexified Lie algebra of K. The pair $(\mathfrak{g},\mathfrak{k})$ is called a *symmetric pair* (of Lie algebras). Then $\mathfrak{g}$ has a nondegenerate symmetric bilinear form B (e.g. the Killing form if $\mathfrak{g}$ is semisimple) and this form is invariant under θ. The (-1)-eigenspace of θ is denoted by $\mathfrak{p}$. It is clear that $\mathfrak{p}$ is a $\mathfrak{k}$-module. Let $\mathfrak{t}$ denote a Cartan subalgebra of $\mathfrak{k}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathfrak{p}$.

Definition 2.6. The Clifford algebra $\mathrm{Cl}(\mathfrak{p})$ is defined with a nondegenerate $\mathfrak{g}$-invariant form B on $\mathfrak{g}$ restricted to $\mathfrak{p}$. We denote by S (or S_1 and S_2 if $\dim \mathfrak{p}$ odd) irreducible (spin) module(s) for $\mathrm{Cl}(\mathfrak{p})$.

2.4. **Types of symmetric pairs.**

Definition 2.7. A symmetric pair $(\mathfrak{g},\mathfrak{k})$ is called

- *primary* if S is $\mathfrak{k}$-primary, i.e., it contains only one $\mathfrak{k}$-type;
- *almost primary* if S has two $\mathfrak{k}$-types and $\dim \mathfrak{a} > 1$;
- *of equal rank* if $\dim \mathfrak{a} = 0$;
- *of almost equal rank* if $\dim \mathfrak{a} = 1$.

The classification of compact symmetric spaces G/K for simple $\mathfrak{g} = \mathrm{Lie}(G)$ is given, for example, in [Hel]. It turns out that all corresponding symmetric pairs of Lie algebras fall into one of the four categories of Definition 2.7 except for the pair $(\mathfrak{e}(6), \mathfrak{sp}(8))$.

In this paper we will consider mostly primary symmetric pairs, such as

$$(\mathfrak{sl}(2n+1), \mathfrak{o}(2n+1)),\ (\mathfrak{sl}(2n), \mathfrak{sp}(2n)),\ (\mathfrak{e}(6), \mathfrak{f}(4)),$$

and almost primary symmetric pairs

$$(\mathfrak{sl}(2n), \mathfrak{o}(2n)).$$

2.5. **Algebra structure of $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ as endomorphisms of spin modules.** Recall that, if $\dim \mathfrak{p}$ is even, the Clifford algebra is isomorphic to the endomorphisms of the spinor module and if $\dim \mathfrak{p}$ is odd, equal to the direct sum of two such spaces.

$$\mathrm{Cl}(\mathfrak{p}) = \begin{cases} \mathrm{End}(S) & \dim \mathfrak{p} \text{ even}, \\ \mathrm{End}(S_1) \oplus \mathrm{End}(S_2) & \dim \mathfrak{p} \text{ odd}. \end{cases}$$

Lemma 2.8. *The algebra* $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ *is equal to the* $\mathfrak{k}$*-endomorphisms of* S *(resp. the direct sum of* $\mathfrak{k}$*-endomorphisms of* S_1 *and* S_2*).*

Proof. If $\dim \mathfrak{p}$ is even then $\mathrm{Cl}(\mathfrak{p}) = \mathrm{End}(S)$ and $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} = \mathrm{End}_{\mathfrak{k}}(S)$ Analogously, for $\dim \mathfrak{p}$ odd, $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} = \mathrm{End}_{\mathfrak{k}}(S_1) \oplus \mathrm{End}_{\mathfrak{k}}(S_2)$. □

Lemma 2.9. [HP2, §2.3.6] *The highest weights of the spin module* S *(resp.* S_1 *and* S_2*) are given by*

$$w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}} \quad \text{for } w \in W^1.$$

Let $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{q}$ where $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^- = \mathfrak{n}^+ \cap \mathfrak{p} \oplus \mathfrak{n}^- \cap \mathfrak{p}$. Define E_w to be the irreducible finite dimensional $\mathfrak{k}$-module with highest weight $w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$. Let $\mathfrak{a}^+, \mathfrak{a}^-$ be a dual pair of maximal isotropic subspaces of $\mathfrak{a}$ with respect to B. If $\mathfrak{a}$ or equivalently $\mathfrak{p}$ is odd dimensional, let $\mathfrak{a}_0$ be the 1-dimensional space orthogonal to $\mathfrak{a}^+ \oplus \mathfrak{a}^-$. Then for $\dim \mathfrak{p}$ even, S is isomorphic to $\bigwedge \mathfrak{a}^+ \otimes \bigwedge \mathfrak{q}^+$. For $\dim \mathfrak{p}$ odd, S_1, S_2 are both isomorphic to $\bigwedge \mathfrak{a}^+ \otimes \bigwedge \mathfrak{q}^+$, with $\mathfrak{a}_0$ acting on S_1^{even}, S_2^{odd} by i and S_1^{odd}, S_2^{even} by $-i$.

Lemma 2.10. [HP2, §2.3.6] *As* $\mathfrak{k}$*-modules,*

$$S \cong \bigoplus_{w \in W^1} \bigwedge \mathfrak{a}^+ \otimes E_w, \quad S_1 \cong S_2 \cong \bigoplus_{w \in W^1} \bigwedge \mathfrak{a}^+ \otimes E_w,$$

with trivial $\mathfrak{k}$*-action on* $\bigwedge \mathfrak{a}^+$*.*

Definition 2.11. Define an algebra $\mathrm{Pr}(S)$ isomorphic to $\mathbb{C}^{|W^1|}$ with basis of idempotents labeled by pr_w, $w \in W^1$, such that

$$\mathrm{pr}_{w_1} \mathrm{pr}_{w_2} = \delta_{w_1, w_2} \mathrm{pr}_{w_1}.$$

Lemma 2.12. *The* $\mathfrak{k}$*-invariants in* $\mathrm{Cl}(\mathfrak{p})$ *are isomorphic as an algebra to*

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong \begin{cases} \mathrm{End}(\bigwedge \mathfrak{a}^+) \otimes \mathrm{Pr}(S) & \dim \mathfrak{p} \textit{ even}, \\ (\mathrm{End}(\bigwedge \mathfrak{a}^+) \oplus \mathrm{End}(\bigwedge \mathfrak{a}^+)) \otimes \mathrm{Pr}(S) & \dim \mathfrak{p} \textit{ odd}. \end{cases}$$

In both cases

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong \mathrm{Cl}(\mathfrak{a}) \otimes \mathrm{Pr}(S).$$

Moreover, $\mathrm{Pr}(S) = \mathrm{im}\, \alpha_Z$*.*

Proof. The algebra $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ is the $\mathfrak{k}$-endomorphisms of S or S_1 and S_2. For each $\mathfrak{k}$-type E_w there is a projection pr_w onto the E_w isotypic component. Furthermore, this isotypic component is a direct sum of $\dim \bigwedge \mathfrak{a}^+$ copies of E_w, hence the space of $\mathfrak{k}$-endomorphisms of this isotypic component is isomorphic to $\mathrm{End}(\bigwedge \mathfrak{a}^+)$. An identical argument can be made for S_1 or S_2. To finish, one notes that $\mathfrak{q}$ is always even dimensional as it contains both positive and negative root spaces. Hence $\dim \mathfrak{p}$ has the same parity as $\dim \mathfrak{a}$. Therefore, if $\dim \mathfrak{p}$ is even, then $\mathrm{Cl}(\mathfrak{a}) = \mathrm{End}(\bigwedge \mathfrak{a}^+)$ and if $\dim \mathfrak{p}$ is odd, then $\mathrm{Cl}(\mathfrak{a}) = \mathrm{End}(\bigwedge \mathfrak{a}^+) \oplus \mathrm{End}(\bigwedge \mathfrak{a}^+)$.

The fact that $\mathrm{Pr}(S) = \mathrm{im}\, \alpha_Z$ was proved in [CGKP]. □

It is worth noting this is not a filtered algebra isomorphism if we use the usual filtration on $\mathrm{Cl}(\mathfrak{a})$. A description of the filtration on $\mathrm{Cl}(\mathfrak{a})$ which is compatible with the above isomorphism will be given in Section 8.

3. Transgressions in Weil algebras

In this section we define various Weil algebras and construct the corresponding transgression maps. The results of this section will allow us to reprove Kostant's description of $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ as the Clifford algebra on primitives. Furthermore, we are able to extend our results to the relative case which will be illuminated in section 7. Theorem 3.26 shows that the image of the transgression map consists of primitives as defined in [Ras]. It turns out that the transgression maps in various Weil algebras all factor through the noncommutative Weil algebra.

3.1. ***G*-differential algebras.** H. Cartan introduced the notion of $\mathfrak{g}$-differential algebras as a generalisation of algebras of differential forms on manifolds with $\mathfrak{g}$-action, [Car1, Car2]. Later $\mathfrak{g}$-differential algebras appeared in the study of equivariant cohomology [GS, AM1, AM2], in Chern–Weil theory [AM3, Mei], and in relation to (algebraic) Dirac operators and Vogan's conjecture [HP1, HP2, AM3]. Here we review definition and main properties of G-differential algebras; see [GS, §2-§4] and [Mei, §6] for more details.

Let $\mathfrak{g}$ be a Lie algebra and ξ be a generator of the Grassmann algebra $\bigwedge[\xi]$, i.e., $\xi^2 = 0$. Let d denote the differential on $\bigwedge[\xi]$ given by $\mathrm{d} = \frac{\partial}{\partial \xi} \in \mathrm{Der} \bigwedge[\xi]$. Consider a semisimple $\mathbb{Z}$-graded Lie superalgebra

$$\widehat{\mathfrak{g}} = \bigwedge[\xi] \otimes \mathfrak{g} \oplus \mathbb{C}\,\mathrm{d} = \widehat{\mathfrak{g}}_{-1} \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1,$$

where $\widehat{\mathfrak{g}}_{-1} = \mathrm{span}(\iota_x \mid x \in \mathfrak{g})$, $\widehat{\mathfrak{g}}_0 = \mathrm{span}(L_x \mid x \in \mathfrak{g})$, and $\widehat{\mathfrak{g}}_1 = \mathrm{span}(\mathrm{d})$. In the following we refer to ι_x as contractions and L_x as Lie derivatives. The explicit brackets in $\widehat{\mathfrak{g}}$ are as follows

$$[L_x, L_y] = L_{[x,y]}, \qquad [L_x, \iota_y] = \iota_{[x,y]}, \qquad [\iota_x, \mathrm{d}] = L_x,$$
$$[\mathrm{d}, \mathrm{d}] = 0, \qquad [L_x, \mathrm{d}] = 0, \qquad [\iota_x, \iota_y] = 0,$$

for all $x, y \in \mathfrak{g}$.

A (graded, filtered) $\mathfrak{g}$-*differential space* is a (graded, filtered) super vector space V, together with a structure of $\widehat{\mathfrak{g}}$-representation $\rho\colon \widehat{\mathfrak{g}} \to \mathfrak{gl}(V)$ (compatible with gradation/filtration). A $\mathfrak{g}$-differential algebra is a super-algebra $\mathcal{A}$, together with a structure of a $\mathfrak{g}$-differential space, such that ρ takes values in $\mathrm{Der}(\mathcal{A})$.

Similarly, let G be a compact Lie group and denote by $\mathfrak{g}$ its Lie algebra. A G-*differential algebra* is a superalgebra $\mathcal{A}$ with a representation ρ of G by automorphisms of $\mathcal{A}$ and a structure of $\mathfrak{g}$-differential algebra on $\mathcal{A}$ which satisfies

$$\frac{d}{dt}\rho(\exp tx)\Big|_{t=0} = L_x, \qquad \rho(g) L_x \rho(g^{-1}) = L_{\mathrm{Ad}_g x},$$
$$\rho(g)\iota_x\rho(g^{-1}) = \iota_{\mathrm{Ad}_g x}, \qquad \rho(g)\,\mathrm{d}\,\rho(g^{-1}) = \mathrm{d},$$

for all $g \in G$ and $x \in \mathfrak{g}$, where Ad denotes the adjoint action of G on $\mathfrak{g}$.

Let $\mathcal{A}$ be a G-differential algebra. The G-horizontal subalgebra of $\mathcal{A}$, denoted by $\mathcal{A}_{G-\mathrm{hor}}$, by

$$\mathcal{A}_{G-\mathrm{hor}} = \{a \in \mathcal{A} \mid \iota_x a = 0 \quad \text{for all } x \in \mathfrak{g}\}.$$

Let us emphasise that $\mathcal{A}_{G-\mathrm{hor}}$ is not a differential subalgebra.

The G-basic subalgebra of $\mathcal{A}$, denoted by $\mathcal{A}_{G-\mathrm{bas}}$ is the algebra of G-invariants in $\mathcal{A}$ which are horizontal, i.e. annihilated by all the contractions by the elements of $\mathfrak{g}$. In other words:

$$\mathcal{A}_{G-\mathrm{bas}} = \mathcal{A}^G \cap \{a \in \mathcal{A} \mid \iota_x a = 0, \forall x \in \mathfrak{g}\}.$$

Similarly, the $\mathfrak{g}$-basic subalgebra of $\mathcal{A}$ is defined by

$$\mathcal{A}_{\mathfrak{g}-\mathrm{bas}} = \mathcal{A}^{\mathfrak{g}} \cap \{a \in \mathcal{A} \mid \iota_x a = 0, \forall x \in \mathfrak{g}\}.$$

Note that $\mathcal{A}_{G-\mathrm{bas}} \subseteq \mathcal{A}_{\mathfrak{g}-\mathrm{bas}}$.

A *connection* on a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ is an odd linear map $\vartheta\colon \mathfrak{g}^* \to \mathcal{A}$ of degree 1, such that $L_x\vartheta(\mu) = \vartheta(\mathrm{ad}_x^*\mu)$ and $\iota_x\vartheta(\mu) = \langle \mu, x\rangle$ for all $\mu \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$. A $\mathfrak{g}$-differential algebra admitting a connection is called *locally free*. If e_i is a basis of $\mathfrak{g}$ then a connection can be represented by the element $\vartheta = \sum_i^{(\mathfrak{g})} \vartheta^i \otimes e_i \in \mathcal{A} \otimes \mathfrak{g}$.

The *curvature of a connection* ϑ is an even map $F^\vartheta\colon \mathfrak{g}^* \to \mathcal{A}$ of degree 2, defined by

$$F^\vartheta = \mathrm{d}\,\vartheta + \frac{1}{2}[\vartheta, \vartheta]. \tag{3.1}$$

The curvature map can be represented by the element $F^\vartheta = \sum_i^{(\mathfrak{g})} (F^\vartheta)^i \otimes e_i \in \mathcal{A} \otimes \mathfrak{g}$, where

$$(F^\vartheta)^i = \mathrm{d}\,\vartheta^i + \frac{1}{2}\sum_{a,b}^{(\mathfrak{g})} c^i_{a,b}\vartheta^a\vartheta^b,$$

where $c^i_{a,b}$ are the structure constants of $\mathfrak{g}$ in the basis e_a, i.e., $[e_a, e_b] = \sum_i c^i_{a,b} e_i$. It is easy to see that the curvature map is $\mathfrak{g}$-equivariant and takes values in $\mathcal{A}_{\mathfrak{g}-\mathrm{hor}}$.

Example 3.2. Consider the case of $\mathcal{A} = \bigwedge \mathfrak{g}^*$. Define contractions ι_x of $\bigwedge \mathfrak{g}^*$ by an element $x \in \mathfrak{g}$ by Definition 2.1. Let L_x denote the extension of the coadjoint action ad_x^* on $\bigwedge \mathfrak{g}^*$ for $x \in \mathfrak{g}$. Let $\mathrm{d}_\wedge\colon \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ be the dual map to the bracket $[-,-]\colon \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$ extended to a degree 1 odd derivation of $\bigwedge \mathfrak{g}^*$; explicitly

$$\mathrm{d}_\wedge = \frac{1}{2}\sum_a^{(\mathfrak{g})} f_a \circ L_{e_a},$$

where e_a is a basis of $\mathfrak{g}$ and f_a is the corresponding dual basis of $\mathfrak{g}^*$ considered as the left multiplication operator. It is easy to see that $\bigwedge \mathfrak{g}^*$ is a $\mathfrak{g}$-differential algebra. The $\mathfrak{g}$-differential algebra $\bigwedge \mathfrak{g}^*$ is locally free with connection $\vartheta_\wedge$ given by $\vartheta_\wedge(\mu) = \mu$ for all $\mu \in \mathfrak{g}^*$. It is easy to see that the curvature of the connection $\vartheta_\wedge$ is zero. See [Mei, §6.7] for further details.

Example 3.3. Assume that $\mathfrak{g}$ admits a non-degenerate symmetric invariant bilinear form B so we can identify $\mathfrak{g}^* \cong \mathfrak{g}$. Consider the corresponding Cartan 3-tensor

$$\phi = \frac{1}{3}\sum_i^{(\mathfrak{g})} \lambda_{\mathfrak{g}}(e_i) \wedge f_i \in (\textstyle\bigwedge \mathfrak{g})^{\mathfrak{g}}. \tag{3.4}$$

The Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ is a filtered $\mathfrak{g}$-differential algebra with differential given by $\mathrm{d}_{\mathrm{Cl}} = [q(\phi), -]$, the Lie derivatives are induced by the adjoint action, and the contractions are defined in §2.2. Clearly, the same formula as for $\bigwedge \mathfrak{g}^*$ defines a flat connection on $\mathrm{Cl}(\mathfrak{g})$.

3.2. **The commutative Weil algebra.** We recall the definition of the (commutative) Weil algebra of $\mathfrak{g}$ and its properties following [Mei, §6]. Let

$$W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \textstyle\bigwedge \mathfrak{g}^*.$$

For $\mu \in \mathfrak{g}^*$, let μ denote generators of $1 \otimes \bigwedge \mathfrak{g}^*$ and $\widehat{\mu}$ denote the generators of $S(\mathfrak{g}^*) \otimes 1$. For $x \in \mathfrak{g}$, let L_x be given by the adjoint action of x and let ι_x be defined as

$$\iota_x := \mathrm{id} \otimes \iota_x,$$

where ι_x in the second factor is as in Example 3.2. Let $\mathrm{d_K}$ denote the Koszul differential relative to the generators μ and $\widehat{\mu}$, that is, $\mathrm{d_K}\,\mu = \widehat{\mu}$, $\mathrm{d_K}\,\widehat{\mu} = 0$. Let $\mathrm{d_{CE}}$ denote the Chevalley–Eilenberg differential given by

$$\mathrm{d_{CE}} = \sum_a^{(\mathfrak{g})} L_{e_a} \otimes f_a + \mathrm{id} \otimes \mathrm{d}_\wedge,$$

where e_a is a basis of $\mathfrak{g}$ and f_a is the corresponding dual basis of $\mathfrak{g}^*$ considered as the left multiplication operator. Then a differential on $W(\mathfrak{g})$ is given by

$$\mathrm{d}_W = \mathrm{d_{CE}} + 2\,\mathrm{d_K}.$$

The graded G-differential algebra $W(\mathfrak{g})$ is called the *commutative Weil algebra* of $\mathfrak{g}$. We have that

$$\mathrm{d}_W(f_i) = 2\widehat{f_i} + \mathrm{d}_\wedge(f_i) = 2\widehat{f_i} - \frac{1}{2}\sum_{a,b}^{(\mathfrak{g})} c^i_{a,b} f_a \wedge f_b.$$

If $\mathfrak{g}$ admits a nondegenerate invariant symmetric bilinear form B, then have that

$$\mathrm{d}_W(\mu) = 2\widehat{\mu} + 2\lambda_{\mathfrak{g}}(\mu) \qquad \text{for } \mu \in \mathfrak{g}^*. \tag{3.5}$$

The Weil algebra is locally free, with connection $\vartheta_W : \mathfrak{g}^* \to W(\mathfrak{g})$ given by $\mu \mapsto \mu$ for $\mu \in \mathfrak{g}^*$. The commutative Weil algebra is universal among (super)commutative $\mathfrak{g}$-differential algebras: for any commutative (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ with connection $\vartheta_{\mathcal{A}}$, there is a unique morphism of (graded, filtered) $\mathfrak{g}$-differential algebras $c_{\mathcal{A}} : W(\mathfrak{g}) \to \mathcal{A}$ such that $c_{\mathcal{A}} \circ \vartheta_W = \vartheta_{\mathcal{A}}$. The homomorphism $c_{\mathcal{A}}$ is called *characteristic homomorphism.* As it was noticed in Example 3.2, the exterior algebra $\bigwedge \mathfrak{g}^*$ is (super)commutative and locally free. Let $\pi_{\wedge\mathfrak{g}^*} : W(\mathfrak{g}) \to \bigwedge \mathfrak{g}^*$ denote the corresponding characteristic homomorphism. On the generators μ and $\widehat{\mu}$ it is given by

$$\pi_{\wedge\mathfrak{g}^*}(\mu) = \mu, \qquad \pi_{\wedge\mathfrak{g}^*}(\widehat{\mu}) = 0.$$

The Weil algebra $W(\mathfrak{g})$ and its $\mathfrak{g}$-invariant part $W(\mathfrak{g})^{\mathfrak{g}}$ are acyclic differential algebras; for example, see [Mei, Proposition 6.9 on p. 149]. Therefore, we can define the *transgression* map in the Weil algebra $W(\mathfrak{g})$ as follows. Since d_W vanishes on $W(\mathfrak{g})_{\mathfrak{g}-\mathrm{bas}} = S(\mathfrak{g}^*)^{\mathfrak{g}}$, any element $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ is a cocycle, hence is a coboundary. A *cochain of transgression* for $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ is an odd element $C_p \in W(\mathfrak{g})^{\mathfrak{g}}$ such that $\mathrm{d}_W(C_p) = p$. For $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ the *transgression* of p is defined by $\mathsf{t}_{\wedge\mathfrak{g}^*}(p) = \pi_{\wedge\mathfrak{g}^*}(C_p)$.

Theorem 3.6 (Transgression theorem)**.** *Let $\mathfrak{g}$ be a complex reductive Lie algebra. Then*

(a) *For $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ the transgression map is independent of the choice of cochain.*
(b) *The transgression map $\mathsf{t}_{\wedge\mathfrak{g}^*}$ satisfies*

$$\ker \mathsf{t}_{\wedge\mathfrak{g}^*} = (S^+(\mathfrak{g}^*)^{\mathfrak{g}})^2, \qquad \mathrm{im}\,\mathsf{t}_{\wedge\mathfrak{g}^*} = P_\wedge(\mathfrak{g}).$$

The construction of the transgression map is due to H. Cartan's article [Car1]. The transgression theorem was also explicitly stated by Chevalley in [Che]; see also [Ler].

3.3. **The noncommutative Weil algebra.** For a $\mathbb{Z}$-graded vector space $E = \bigoplus_k E^k$ and $n \in \mathbb{Z}$ define a graded space with a degree shift $E[n]$ so that $(E[n])^k = E^{n+k}$. In other words, if $x \in E$ is of degree k, then its degree in $E[n]$ is $k-n$. In particular, if E is just a vector space we consider it as graded in degree 0, so $E[n]$ is the same space viewed in degree $n \in \mathbb{Z}$.

Following [AM3], define the *noncommutative Weil* algebra of $\mathfrak{g}$ as the tensor algebra

$$\widetilde{W}(\mathfrak{g}) = T(\mathfrak{g}^*[-2] \oplus \mathfrak{g}^*[-1]).$$

For $\mu \in \mathfrak{g}^*$ denote by μ the corresponding element of $\mathfrak{g}^*[-1]$ and by $\bar{\mu}$ the corresponding element of $\mathfrak{g}^*[-2]$. The algebra $\widetilde{W}(\mathfrak{g})$ has a natural $\mathbb{Z}$-grading, with generators μ being of degree 1 and generators $\bar{\mu}$ of degree 2. In particular, we can view $\widetilde{W}(\mathfrak{g})$ as a $\mathbb{Z}$-graded superalgebra.

The algebra $\widetilde{W}(\mathfrak{g})$ has a natural structure of a graded $\mathfrak{g}$-differential algebra defined as follows; note that by the universal property of the tensor algebra we can define algebra homomorphisms or derivations freely on the generators μ and $\bar{\mu}$.

The differential of $\widetilde{W}(\mathfrak{g})$ is the degree 1 derivation $d_{\widetilde{W}}$ defined on generators by

$$\mathrm{d}_{\widetilde{W}}\, \mu = \bar{\mu}; \qquad \mathrm{d}_{\widetilde{W}}\, \bar{\mu} = 0. \tag{3.7}$$

Contraction by an element $x \in \mathfrak{g}$ is the odd derivation defined on generators by

$$\iota_x \mu = \langle x, \mu \rangle; \qquad \iota_x \bar{\mu} = \mathrm{ad}_x^*\, \mu. \tag{3.8}$$

The Lie derivatives L_x for $x \in \mathfrak{g}$ are the even derivations defined on generators by

$$L_x \mu = \mathrm{ad}_x^*\, \mu; \qquad L_x \bar{\mu} = \mathrm{ad}_x^*\, \bar{\mu} = \overline{\mathrm{ad}_x^*\, \mu}. \tag{3.9}$$

The $\mathfrak{g}$-differential algebra $\widetilde{W}(\mathfrak{g})$ is locally free, with connection $\vartheta_{\widetilde{W}} \colon \mathfrak{g}^* \to \widetilde{W}(\mathfrak{g})$ given by

$$\vartheta_{\widetilde{W}}(\mu) = \mu, \qquad \mu \in \mathfrak{g}^*. \tag{3.10}$$

Let e_i be a basis of $\mathfrak{g}$ and f_i be the corresponding dual basis of $\mathfrak{g}^* \cong \mathfrak{g}^*[-1]$. Since $\vartheta_{\widetilde{W}}$ is the identity map from $\mathfrak{g}^*$ to $\mathfrak{g}^* \cong \mathfrak{g}^*[-1] \subset \widetilde{W}(\mathfrak{g})$, it can be written as

$$\vartheta_{\widetilde{W}} = \sum_i^{(\mathfrak{g})} f_i \otimes e_i \in \widetilde{W} \otimes \mathfrak{g}$$

under the identification $\mathrm{Hom}(\mathfrak{g}^*, \widetilde{W}(\mathfrak{g})) = \widetilde{W}(\mathfrak{g}) \otimes \mathfrak{g}$.

The curvature $F^{\vartheta_{\widetilde{W}}} = \sum_i^{(\mathfrak{g})} (F^\vartheta)^i \otimes e_i \in \widetilde{W}(\mathfrak{g}) \otimes \mathfrak{g}$ of the connection $\vartheta_{\widetilde{W}}$ is given by

$$(F^\vartheta)^i = \mathrm{d}_{\widetilde{W}}\, f_i + \frac{1}{2} \sum_{a,b}^{(\mathfrak{g})} c^i_{a,b} f_a \otimes f_b = \bar{f}_i - \delta(f_i)$$

where $c^i_{a,b} = B([e_a, e_b], f_i)$ are the structure constants of $\mathfrak{g}$ corresponding to the basis e_i, i.e.,

$$[e_a, e_b] = \sum_i c^i_{a,b} e_i,$$

and $\delta : \mathfrak{g}^*[-1] \to \widetilde{W}(\mathfrak{g})$ is the linear map defined by

$$\delta(f_i) = -\frac{1}{2}\sum_{a,b}^{(\mathfrak{g})} c^i_{a,b} f_a \otimes f_b. \tag{3.11}$$

It is easy to see that $(F^\vartheta)^i \in \widetilde{W}(\mathfrak{g})_{\mathfrak{g}-\mathrm{hor}}$. The map δ is analogous to $\mathrm{d}_\wedge$ and satisfies (partial) Cartan calculus, see Lemma 3.16 below. It also satisfies

$$\iota_x \iota_y \delta(\mu) = \langle \mu, [x,y]\rangle, \qquad \mu \in \mathfrak{g}^*[-1],\ x, y \in \mathfrak{g}, \tag{3.12}$$

i.e., it is dual to the Lie bracket of $\mathfrak{g}$.

We define the hat variables, or curvature variables, by

$$\widehat{f}_i := (F^\vartheta)^i = \bar{f}_i - \delta(f_i).$$

Note that $\widehat{f}_i \in \widetilde{W}^2(\mathfrak{g})_{\mathfrak{g}-\mathrm{hor}}$.

Lemma 3.13. *Let W' be the tensor algebra of the vector space spanned by all $\mu, \widehat{\mu}$ for $\mu \in \mathfrak{g}^*$ and define $f \colon W' \to \widetilde{W}(\mathfrak{g})$ by $f(\mu) = \mu$, $f(\widehat{\mu}) = \bar{\mu} - \delta(\mu)$. Then f is an algebra isomorphism.*

Proof. By the universal property of tensor algebra f is an algebra morphism. It is obviously injective. We can construct the inverse homomorphism by putting $f^{-1}(\mu) = \mu$, $f^{-1}(\bar{\mu}) = \widehat{\mu} + \delta(\mu)$ and extending to $\widetilde{W}(\mathfrak{g})$ by the universal property. □

Therefore, μ and $\widehat{\mu}$ form another set of generators of $\widetilde{W}(\mathfrak{g})$. The main advantage of this change of generators is that

$$\iota_x \mu = \langle \mu, x\rangle, \qquad \iota_x \widehat{\mu} = 0, \quad \text{for } x \in \mathfrak{g}.$$

We set $\delta(\widehat{f}_i) = 0$, and extend δ to a degree 1 derivation of $\widetilde{W}(\mathfrak{g})$.

Consider a $\mathbb{Z}^2$-grading on $\widetilde{W}(\mathfrak{g})$

$$\widetilde{W}(\mathfrak{g}) = \bigoplus_{i,j \in \mathbb{N}_0} \widetilde{W}^{i,j}(\mathfrak{g}) \tag{3.14}$$

given by $\deg(\widehat{\mu}) = (1,0)$ and $\deg(\mu) = (0,1)$, then

$$\mathrm{d}_{\widetilde{W}} \colon \widetilde{W}^{i,j}(\mathfrak{g}) \to \widetilde{W}^{i+1,j-1}(\mathfrak{g}) \oplus W^{i,j+1}(\mathfrak{g}). \tag{3.15}$$

Let $\widetilde{W}^{+,0}(\mathfrak{g})$ denote the augmentation ideal in the subalgebra $\widetilde{W}^{\bullet,0}(\mathfrak{g})$ of $\widetilde{W}(\mathfrak{g})$ generated by $\widehat{\mu}$.

The following lemma shows that δ, ι_x and L_x define an analogue of Cartan's calculus on $\widetilde{W}^{0,\bullet}(\mathfrak{g})$. It will be useful in proving the main result of this subsection, Theorem 3.26, but it can also serve as motivation for our definition of δ.

Lemma 3.16. *For $x \in \mathfrak{g}$ we have that on $\widetilde{W}^{0,\bullet}(\mathfrak{g})$*

$$[\delta, L_x] = 0, \qquad [\delta, \iota_x] = L_x, \qquad \iota_x \delta^2 = 0.$$

Proof. 1) Since for $\mu \in \mathfrak{g}^*$ the element $\delta(\mu)$ is an anti-symmetric tensor and the action of $\mathfrak{g}$ preserves anti-symmetric tensors, it is enough show that $\iota_x \circ \iota_y \circ L_z \circ \delta(\mu) = \iota_x \circ \iota_y \circ \delta \circ L_z(\mu)$ for $x, y, z \in \mathfrak{g}$. We have

$$\begin{aligned}
\iota_x \circ \iota_y \circ L_z \circ \delta(\mu) &= \iota_x \circ (L_z \circ \iota_y - \iota_{[z,y]}) \circ \delta(\mu) \\
&= (L_z \circ \iota_x \circ \iota_y - \iota_{[z,x]} \circ \iota_y - \iota_x \circ \iota_{[z,y]}) \circ \delta(\mu) \\
&= L_z \circ \iota_x \circ \iota_y \circ \delta(\mu) - \iota_{[z,x]} \circ \iota_y \circ \delta(\mu) - \iota_x \circ \iota_{[z,y]} \circ \delta(\mu).
\end{aligned}$$

Since $\iota_x \circ \iota_y \circ \delta(\mu) \in \mathbb{C}$, the first term is zero, and using (3.12) we get

$$\begin{aligned} &= -\langle \mu, [[z,x],y] + [x,[z,y]]\rangle = -\langle \mu, [z,[x,y]]\rangle \\ &= \langle L_z\mu, [x,y]\rangle = \iota_x \circ \iota_y \circ \delta \circ L_z(\mu). \end{aligned}$$

2) Let $e_k \in \mathfrak{g}$ and $f_i \in \mathfrak{g}^*$ be dual bases. We have

$$[\iota_{e_k}, \delta](f_i) = \iota_{e_k}\left(-\frac{1}{2}\sum_{a,b} c^i_{a,b} f_a \otimes f_b\right) = -\sum_a c^i_{k,a} f_a = L_{e_k} f_i.$$

The last equality follows from

$$\langle L_{e_k} f_i, e_r\rangle = -\langle f_i, [e_k, e_r]\rangle = -\sum_j \langle f_i, c^j_{k,r} e_j\rangle = -c^i_{k,r} = -\left\langle \sum_a c^i_{k,a} f_a, e_r\right\rangle.$$

The claim follows.

3) We have

$$\begin{aligned} \iota_x \delta^2(\mu) &= (L_x - \delta \circ \iota_x) \circ \delta(\mu) \\ &= L_x \circ \delta(\mu) - \delta \circ (L_x - \delta \circ \iota_x)(\mu) \\ &= L_x \circ \delta(\mu) - L_x \circ \delta(\mu) = 0. \end{aligned}$$

Which concludes the proof. □

As shown in [AM3, §3.3], the noncommutative Weil algebra has the following universal property. Given a (graded, filtered) $\mathfrak{g}$-differential algebra $\mathcal{A}$ with a connection $\vartheta_{\mathcal{A}}$ (such $\mathcal{A}$ is called locally free), there is a unique morphism of (graded, filtered) $\mathfrak{g}$-differential algebras

$$\widetilde{\pi}_{\mathcal{A}} \colon \widetilde{W}(\mathfrak{g}) \to \mathcal{A}, \quad \text{such that} \quad \widetilde{\pi}_{\mathcal{A}} \circ \vartheta_{\widetilde{W}(\mathfrak{g})} = \vartheta_{\mathcal{A}}. \tag{3.17}$$

The morphism $\widetilde{\pi}_{\mathcal{A}}$ is called the *characteristic homomorphism.*

Explicitly, if $\vartheta_{\mathcal{A}} = \sum_i^{(\mathfrak{g})} \vartheta^i_{\mathcal{A}} \otimes e_i \in \mathcal{A} \otimes \mathfrak{g}$, then

$$\widetilde{\pi}_{\mathcal{A}}(f_i) = \vartheta^i_{\mathcal{A}}, \quad \widetilde{\pi}_{\mathcal{A}}(\bar{f}_i) = \widetilde{\pi}_{\mathcal{A}}(\mathrm{d}_{\widetilde{W}}\, f_i) = \mathrm{d}_{\mathcal{A}}\, \vartheta^i_{\mathcal{A}}, \quad \widetilde{\pi}_{\mathcal{A}}(\widehat{f}_i) = \mathrm{d}_{\mathcal{A}}\, \vartheta^i_{\mathcal{A}} + \frac{1}{2}\sum_{a,b}^{(\mathfrak{g})} c^i_{a,b} \vartheta^a_{\mathcal{A}} \vartheta^b_{\mathcal{A}}. \tag{3.18}$$

We will be especially interested in the case when the connection $\vartheta_{\mathcal{A}}$ of the $\mathfrak{g}$-differential algebra $\mathcal{A}$ is flat, i.e., its curvature $F^{\vartheta_{\mathcal{A}}}$ defined by (3.1) vanishes. Then the corresponding characteristic homomorphism $\widetilde{\pi}_{\mathcal{A}} : \widetilde{W}(\mathfrak{g}) \to \mathcal{A}$ has the following properties:

Lemma 3.19. *Let $\mathcal{A}$ be a locally free algebra with a flat connection and $\widetilde{\pi}_{\mathcal{A}} \colon \widetilde{W}(\mathfrak{g}) \to \mathcal{A}$ be the corresponding characteristic homomorphism. Then*

(a) $\widetilde{\pi}_{\mathcal{A}}(\widehat{\mu}) = 0$ *for* $\mu \in \mathfrak{g}^*$,
(b) $\widetilde{\pi}_{\mathcal{A}}(b) = 0$ *for* $b \in \ker \mathrm{d}_{\widetilde{W}}$,
(c) $\widetilde{\pi}_{\mathcal{A}} \circ \delta^2 = 0$.

Proof. (a) Follows from the definition of $\widehat{\mu}$ as curvature components and the fact that the connection on $\mathcal{A}$ is flat. Namely, since $\widetilde{\pi}_{\mathcal{A}}$ is the characteristic homomorphism, it is a morphism of $\mathfrak{g}$-differential algebras that agrees with connections:

$\widetilde{\pi}_{\mathcal{A}}(\vartheta^i_{\widetilde{W}}) = \vartheta^i_{\mathcal{A}}$. Hence, the curvature of the connection on $\widetilde{W}(\mathfrak{g})$ maps to the curvature of the connection on $\mathcal{A}$:

$$\begin{aligned}\widetilde{\pi}_{\mathcal{A}}((F^{\vartheta_{\widetilde{W}}})^i) &= \widetilde{\pi}_{\mathcal{A}}\left(\mathrm{d}_{\widetilde{W}}\,\vartheta^i_{\widetilde{W}} + \frac{1}{2}\sum_{a,b} c^i_{a,b}\vartheta^a_{\widetilde{W}}\otimes\vartheta^b_{\widetilde{W}}\right)\\ &= \mathrm{d}_{\mathcal{A}}(\widetilde{\pi}_{\mathcal{A}}(\vartheta^i_{\widetilde{W}})) + \frac{1}{2}\sum_{a,b} c^i_{a,b}\widetilde{\pi}_{\mathcal{A}}(\vartheta^a_{\widetilde{W}})\widetilde{\pi}_{\mathcal{A}}(\vartheta^b_{\widetilde{W}})\\ &= \mathrm{d}_{\mathcal{A}}\,\vartheta^i_{\mathcal{A}} + \frac{1}{2}\sum_{a,b} c^i_{a,b}\vartheta^a_{\mathcal{A}}\vartheta^b_{\mathcal{A}} = (F^{\vartheta_{\mathcal{A}}})^i.\end{aligned}$$

(b) From (3.15) it is easy to see that $\ker \mathrm{d}_{\widetilde{W}} \cap \widetilde{W}^{0,+}(\mathfrak{g}) = \{0\}$.

(c) First note that

$$0 = \widetilde{\pi}_{\mathcal{A}}(\widehat{\mu}) = \widetilde{\pi}_{\mathcal{A}}(\bar{\mu} - \delta(\mu)) = \widetilde{\pi}_{\mathcal{A}}(\mathrm{d}_{\widetilde{W}}\,\mu - \delta(\mu)) = \mathrm{d}_{\mathcal{A}} \circ \widetilde{\pi}_{\mathcal{A}}(\mu) - \widetilde{\pi}_{\mathcal{A}} \circ \delta(\mu).$$

Therefore

$$\widetilde{\pi}_{\mathcal{A}} \circ \delta^2(\mu) = \mathrm{d}_{\mathcal{A}} \circ \widetilde{\pi}_{\mathcal{A}} \circ \delta(\mu) = \mathrm{d}^2_{\mathcal{A}} \circ \widetilde{\pi}_{\mathcal{A}}(\mu) = 0.$$

This completes the proof. □

We would like to define a transgression map for $\widetilde{W}(\mathfrak{g})$ in a similar manner as we did for the commutative Weil algebra $W(\mathfrak{g})$ in Subsection 3.2 (see Theorem 3.6 and the text above it.) We would however like this map to have values in an arbitrary $\mathfrak{g}$-differential algebra $\mathcal{A}$ with a flat connection. It turns out we can not define such a map on all of $\widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}} := T^+(\mathfrak{g}[-2])^{\mathfrak{g}} \otimes 1 \subset \widetilde{W}(\mathfrak{g})$, but on a certain subspace that we call the transgression space, defined below.

To define the transgression space, we first recall the symmetrization map

$$\mathrm{sym}_W \colon W(\mathfrak{g}) \to \widetilde{W}(\mathfrak{g})$$

described in [Mei, Proposition 6.13]. This map is obtained by symmetrizing the (even) variables $\bar{\mu}$ and skew-symmetrizing the (odd) variables μ. Explicitly, sym_W is the linear map defined on monomials $v_1 \dots v_k$ in $W(\mathfrak{g})$ such that each of $v_1, \dots, v_k$ is either some μ or some $\bar{\mu}$, by

$$\mathrm{sym}_W(v_1 \dots v_k) = \frac{1}{k!}\sum_{\sigma\in\mathbf{S}_k}(-1)^{N_\sigma(v_1,\dots,v_k)}\phi(v_{\sigma^{-1}(1)})\dots\phi(v_{\sigma^{-1}(k)}),$$

where $\mathbf{S}_k$ is the symmetric group, and $N_\sigma(v_1, \dots, v_k)$ is the number of pairs $i < j$ such that v_i, v_j are odd elements and $\sigma^{-1}(i) > \sigma^{-1}(j)$.

It is proved in [Mei, Proposition 6.13] that sym_W is an isomorphism of $\mathfrak{g}$-differential spaces. (It is not an algebra homomorphism, but it is an algebra morphism up to homotopy; see [AM3].) From this and the above definition, it is clear that

$$\begin{aligned}&\mathrm{sym}_W(f_i) = f_i, \qquad \mathrm{sym}_W(\mathrm{d}_W\, f_i) = \mathrm{d}_{\widetilde{W}}\, f_i = \bar{f}_i,\\ &\mathrm{sym}_W(\mathrm{d}_\wedge(f_i)) = \delta(f_i),\\ &\mathrm{sym}_W(\widehat{f}_i) = \frac{1}{2}\mathrm{sym}_W(\mathrm{d}_W\, f_i - \mathrm{d}_\wedge(f_i)) = \frac{1}{2}(\mathrm{d}_{\widetilde{W}}\, f_i - \delta(f_i)) = \frac{1}{2}\widehat{f}_i;\end{aligned}$$

for the last row, see (3.5).

If $p \in S(\mathfrak{g}^*)^{\mathfrak{g}} \subset W(\mathfrak{g})$, then $\mathrm{d}_W p = 0$, hence $\mathrm{d}_{\widetilde{W}}(\mathrm{sym}_W(p)) = 0$. Moreover, since $W(\mathfrak{g})$ is acyclic, we also have that $\mathrm{sym}_W(p) \in \mathrm{im}\,\mathrm{d}_{\widetilde{W}}$ if $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$. This motivates the following definition.

Definition 3.20. The *transgression space* for $\widetilde{W}(\mathfrak{g})$ is

$$\widetilde{\mathfrak{T}}(\mathfrak{g}) = \{\mathrm{sym}_W(p) \in \widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}} \mid p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}\} \subset \mathrm{im}\,\mathrm{d}_{\widetilde{W}}\,.$$

We note that $\widetilde{\mathfrak{T}}(\mathfrak{g})$ is a graded subspace of $\widetilde{W}(\mathfrak{g})$, and that by the above remarks, it is contained in $\mathrm{im}\,\mathrm{d}_{\widetilde{W}}$.

We now define the transgression map for the non-commutative Weil algebra as follows.

Definition 3.21. Let $\mathcal{A}$ be a locally free $\mathfrak{g}$-differential algebra with a flat connection. The (universal) *transgression* map of $\widetilde{W}(\mathfrak{g})$ with values in $\mathcal{A}^{\mathfrak{g}}$ is the linear map $\widetilde{\mathfrak{t}}_{\mathcal{A}} \colon \widetilde{\mathfrak{T}}(\mathfrak{g}) \to \mathcal{A}^{\mathfrak{g}}$ defined by

$$\widetilde{\mathfrak{t}}_{\mathcal{A}}(p) = \widetilde{\pi}_{\mathcal{A}}(\widetilde{C}_p),$$

where $\widetilde{C}_p \in \widetilde{W}(\mathfrak{g})^{\mathfrak{g}}$ is such that $\mathrm{d}_{\widetilde{W}}\, \widetilde{C}_p = p$. (Any such $\widetilde{C}_p$ is called a *cochain of transgression* for p. It always exists, since $\widetilde{\mathfrak{T}}(\mathfrak{g}) \subset \mathrm{im}\,\mathrm{d}_{\widetilde{W}}$.)

Lemma 3.22. *The transgression map $\widetilde{\mathfrak{t}}_{\mathcal{A}}$ is well-defined.*

Proof. Assume $p = \mathrm{d}_{\widetilde{W}}\, C = \mathrm{d}_{\widetilde{W}}\, C'$. Then $\mathrm{d}_{\widetilde{W}}(C - C') = 0$, so $(C - C') \in \ker \mathrm{d}_{\widetilde{W}}$. Thus $\widetilde{\pi}_{\mathcal{A}}(C - C') = 0$ by Lemma 3.19(b). Therefore, the transgression map does not depend on a choice of cochain of transgression. □

Let $\Phi \colon \widetilde{W}(\mathfrak{g}) \to \widetilde{W}(\mathfrak{g})$ be the $\mathfrak{g}$-equivariant algebra map defined on generators by

$$\Phi(\mu) = \mu, \qquad \Phi(\widehat{\mu}) = \delta(\mu). \tag{3.23}$$

Define the $\mathfrak{g}$-equivariant algebra homomorphism

$$\Phi_{\mathcal{A}} := \widetilde{\pi}_{\mathcal{A}} \circ \Phi : \widetilde{W}(\mathfrak{g}) \to \mathcal{A}. \tag{3.24}$$

Furthermore, let $\iota_{\widehat{x}}$, $x \in \mathfrak{g}$ be the derivation of $\widetilde{W}(\mathfrak{g})$ of degree -2 defined on generators by

$$\iota_{\widehat{x}}(\mu) = 0, \qquad \iota_{\widehat{x}}(\widehat{\mu}) = \langle x, \mu\rangle. \tag{3.25}$$

The main result of this subsection is the following theorem; it is a generalisation of Theorem 73 in [Kos2], and its proof is a generalisation of [Mei, Proposition 6.19 on p. 160].

This result will be used in Subsection 4 to get an analogous relative version, which will in turn be essential for the proof of our main result, Theorem 1.1.

Theorem 3.26. *If $x \in \mathfrak{g}$ and $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$, then*

$$\iota_x \widetilde{\mathfrak{t}}_{\mathcal{A}}(p) = \frac{(m!)^2}{(2m)!} \cdot \Phi_{\mathcal{A}}(\iota_{\widehat{x}} p).$$

Proof. The first step is to rewrite the map $\widetilde{\mathfrak{t}}_{\mathcal{A}}$ more explicitly, using a homotopy map which will send any $p \in \widetilde{\mathfrak{T}}(\mathfrak{g})$ to a cochain of transgression for p.

To define this homotopy map, we consider $\mathbb{C}[t, dt]$, the graded commutative differential algebra with an even generator t of degree 0 and an odd generator dt of degree 1, and with a single relation $(dt)^2 = 0$. The differential on $\mathbb{C}[t, dt]$ is defined by $\mathrm{d}(t) = dt$, $\mathrm{d}(dt) = 0$. One can think of $\mathbb{C}[t, dt]$ as an algebraic counterpart to differential forms on the unit interval.

Define an "integration operator" $J\colon \mathbb{C}[t, dt] \to \mathbb{C}$ by

$$J\left(\sum_{k\geq 0} a_k t^k + \sum_{l\geq 0} b_l t^l dt\right) = \sum_{l\geq 0} \frac{b_l}{l+1} \qquad \text{for } a_k, b_l \in \mathbb{C}. \tag{3.27}$$

In other words, if P and Q are polynomials in t, then

$$J(P + Qdt) = \int_0^1 Qdt.$$

Let $\varepsilon\colon \widetilde{W}(\mathfrak{g}) \to \mathbb{C}$ be the augmentation map and $i\colon \mathbb{C} \to \widetilde{W}(\mathfrak{g})$ be the unit map. The standard homotopy operator h between id and $i \circ \varepsilon$ on $\widetilde{W}(\mathfrak{g})$ is defined by

$$h = (J \otimes \mathrm{id}) \circ \phi_h, \tag{3.28}$$

where ϕ_h is the differential algebra morphism from $\widetilde{W}(\mathfrak{g})$ to $\mathbb{C}[dt, t] \otimes \widetilde{W}(\mathfrak{g})$ defined on generators by

$$\phi_h(\mu) = t\mu, \qquad \phi_h(\bar{\mu}) = dt\mu + t\bar{\mu}. \tag{3.29}$$

Lemma 3.30. *Let h be the homotopy map defined above, and let i respectively ε be the unit and the counit map of $\widetilde{W}(\mathfrak{g})$. Then*

$$h\,\mathrm{d}_{\widetilde{W}} + \mathrm{d}_{\widetilde{W}}\,h = \mathrm{id} - i \circ \varepsilon.$$

Moreover, h commutes with L_x for all $x \in \mathfrak{g}$ and $h(p) \in \widetilde{W}(\mathfrak{g})^{\mathfrak{g}}$.

Proof. This is analogous to the commutative case, see [Mei, Section 6.4]. □

Lemma 3.31. *Let h be the homotopy operator defined above. Then $\widetilde{\mathfrak{t}}_{\mathcal{A}} = \widetilde{\pi}_{\mathcal{A}} \circ h$.*

Proof. Note that for $p \in \widetilde{\mathfrak{T}}(\mathfrak{g})$ we have that $i \circ \varepsilon(p) = 0$ and $\mathrm{d}_{\widetilde{W}}(p) = 0$. Therefore, it follows from Lemma 3.30 that

$$\mathrm{d}_{\widetilde{W}}(h(p)) = p.$$

Hence we can take $h(p)$ as a cochain of transgression for p. □

We now get back to the proof of Theorem 3.26. Our next task is to write the left side of the claimed equality using Lemma 3.30. Let g be the derivation of $\widetilde{W}(\mathfrak{g})$ defined on generators by

$$g(\widehat{\mu}) = \mu, \qquad g(\mu) = 0. \tag{3.32}$$

Proposition 3.33. *For any $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$,*

$$\widetilde{\mathfrak{t}}_{\mathcal{A}}(p) = \frac{(m!)^2}{(2m+1)!} \Phi_{\mathcal{A}}(g(p)).$$

Proof. Since p is in $\widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}) \subseteq \widetilde{W}^{+,0}(\mathfrak{g})^{\mathfrak{g}}$, we can write it as

$$p = \sum \widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_{m+1}.$$

It is easy to see that $\widetilde{\mathfrak{t}}_{\mathcal{A}} = \widetilde{\pi}_{\mathcal{A}} \circ h = (J \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \widetilde{\pi}_{\mathcal{A}}) \circ \phi_h$. Note that

$$\phi_h(\widehat{\mu}_i) = \phi_h(\bar{\mu}_i - \Phi(\widehat{\mu}_i)) = dt\mu_i + t\bar{\mu}_i - t^2\Phi(\widehat{\mu}_i) = dt\mu_i + t\widehat{\mu}_i + (t - t^2)\Phi(\widehat{\mu}_i),$$

using the fact that $\Phi(\widehat{\mu}_i) = \delta(\mu_i)$, and that it is in $\widetilde{W}^{0,2}(\mathfrak{g})$, so ϕ_h acts on $\Phi(\widehat{\mu}_i)$ by multiplying by t^2. This implies

$$\begin{aligned}(J\otimes \mathrm{id})\circ(\mathrm{id}\otimes\widetilde{\pi}_{\mathcal{A}})\circ\phi_h(p) = \\ = \sum (J\otimes \mathrm{id})\circ(\mathrm{id}\otimes\widetilde{\pi}_{\mathcal{A}})\Big((dt\mu_1 + t\widehat{\mu}_1 + (t-t^2)\Phi(\widehat{\mu}_1))\cdot\ldots \\ \ldots\cdot(dt\mu_{m+1} + t\widehat{\mu}_{m+1} + (t-t^2)\Phi(\widehat{\mu}_{m+1}))\Big).\end{aligned}$$

Applying $\widetilde{\pi}_{\mathcal{A}}$ and remembering that $\widetilde{\pi}_{\mathcal{A}}(\widehat{\mu}_i) = 0$ by Lemma 3.19, we find

$$\begin{aligned}(J\otimes \mathrm{id})\circ(\mathrm{id}\otimes\widetilde{\pi}_{\mathcal{A}})\circ\phi_h(p) = \\ = \sum (J\otimes \mathrm{id})\Big((dt\widetilde{\pi}_{\mathcal{A}}(\mu_1) + (t-t^2)\Phi_{\mathcal{A}}(\widehat{\mu}_1))\cdot\ldots \\ \ldots\cdot(dt\widetilde{\pi}_{\mathcal{A}}(\mu_{m+1}) + (t-t^2)\Phi_{\mathcal{A}}(\widehat{\mu}_{m+1}))\Big).\end{aligned}$$

which has linear dt term

$$(J\otimes \mathrm{id})\sum (t-t^2)^m dt \sum_i \Phi_{\mathcal{A}}(\widehat{\mu}_1\otimes\cdots\otimes\widehat{\mu}_{i-1})\widetilde{\pi}_{\mathcal{A}}(\mu_i)\Phi_{\mathcal{A}}(\widehat{\mu}_{i+1}\otimes\cdots\otimes\widehat{\mu}_{m+1}).$$

The lemma follows by noting that $J\left((t-t^2)^m dt\right) = \frac{(m!)^2}{(2m+1)!}$. □

Getting back to the proof of Theorem 3.26, we now see that the left side of the claimed equality is equal to

$$\frac{(m!)^2}{(2m+1)!}\iota_x\Phi_{\mathcal{A}}(g(p)) = \frac{(m!)^2}{(2m+1)!}\widetilde{\pi}_{\mathcal{A}}(\iota_x\Phi(g(p))).$$

On the other hand, the right side of the equality we need to prove is $\frac{(m!)^2}{(2m)!}\Phi_{\mathcal{A}}(\iota_{\widehat{x}}p)$, so the equality to prove becomes

$$\widetilde{\pi}_{\mathcal{A}}(\iota_x\Phi(g(p))) = (2m+1)\Phi_{\mathcal{A}}(\iota_{\widehat{x}}p). \tag{3.34}$$

Now we rewrite the right side of (3.34) as follows. Let $\eta : \widetilde{W}(\mathfrak{g}) \to \widetilde{W}(\mathfrak{g})$ be a derivation of degree 0 defined on generators by

$$\eta(\mu) = 0, \qquad \eta(\widehat{\mu}) = \delta(\mu), \quad \mu\in\mathfrak{g}^*. \tag{3.35}$$

We claim that for any $q\in\widetilde{W}^{m,0}(\mathfrak{g})$ we have

$$\Phi_{\mathcal{A}}(q) = \widetilde{\pi}_{\mathcal{A}}\left(\frac{\eta^m}{m!}(q)\right). \tag{3.36}$$

In particular, this will hold for $q = \iota_{\widehat{x}}p \in \widetilde{W}^{m,0}(\mathfrak{g})$.

To prove (3.36), we can assume $q = \widehat{\mu}_1\otimes\cdots\otimes\widehat{\mu}_m$ and write

$$\Phi_{\mathcal{A}}(q) = \widetilde{\pi}_{\mathcal{A}}(\Phi(\widehat{\mu}_1)\otimes\cdots\otimes\Phi(\widehat{\mu}_m)) = \widetilde{\pi}_{\mathcal{A}}(\delta(\mu_1)\otimes\cdots\otimes\delta(\mu_m)) = \widetilde{\pi}_{\mathcal{A}}(\eta(\widehat{\mu}_1)\otimes\cdots\otimes\eta(\widehat{\mu}_m)).$$

On the other hand, by Lemma 3.19 $\widetilde{\pi}_{\mathcal{A}}$ kills all $\widehat{\mu}_i$, as well as all $\delta^2(\mu_i) = \eta^2(\widehat{\mu}_i)$. It follows that

$$\widetilde{\pi}_{\mathcal{A}}\left(\frac{\eta^m}{m!}(q)\right) = \widetilde{\pi}_{\mathcal{A}}(\eta(\widehat{\mu}_1)\otimes\cdots\otimes\eta(\widehat{\mu}_m)).$$

This proves (3.36). In particular, (3.36) for $q = \iota_{\widehat{x}}p$ implies that the right side of (3.34) is equal to

$$\frac{2m+1}{m!}\widetilde{\pi}_{\mathcal{A}}(\eta^m(\iota_{\widehat{x}}p)) = \frac{2m+1}{m!}\widetilde{\pi}_{\mathcal{A}}(\iota_{\widehat{x}}\eta^m(p)); \tag{3.37}$$

the last equality follows from the fact that $\iota_{\widehat{x}}\eta = \eta\iota_{\widehat{x}}$, which is clear since both $\iota_{\widehat{x}}\eta$ and $\eta\iota_{\widehat{x}}$ are 0 on generators μ and $\widehat{\mu}$.

To pass between $\iota_{\widehat{x}}$ and ι_x, we need the following lemma. Let $\varphi : \widetilde{W}(\mathfrak{g}) \to \widetilde{W}(\mathfrak{g})$ be the linear map acting by 0 on the constants, and by $\frac{1}{i+j}$ on the subspace $\widetilde{W}^{i,j}(\mathfrak{g}) \subset \widetilde{W}(\mathfrak{g})$ consisting of elements of degree i with respect to the hat variables and of degree j with respect to the μ variables, with $i+j>0$.

Lemma 3.38. *Let* $p = \sum \widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_{m+1} \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$. *Then*

$$\widetilde{\pi}_{\mathcal{A}}(\iota_{\widehat{x}}\eta^m(p)) = \widetilde{\pi}_{\mathcal{A}}(\iota_x \circ g \circ \varphi(\eta^m(p))).$$

Assuming Lemma 3.38, we now finish the proof of Theorem 3.26. Recall that we need to prove (3.34), and that we have rewritten the right side of (3.34) in (3.37). Using Lemma 3.38, (3.37) becomes

$$\frac{2m+1}{m!}\widetilde{\pi}_{\mathcal{A}}(\iota_x \circ g \circ \varphi(\eta^m(p))),$$

which is equal to $\frac{1}{m!}\widetilde{\pi}_{\mathcal{A}}(\iota_x \circ g(\eta^m(p)))$ since $\eta^m(p)$ has bidegree $(1, 2m)$ so φ acts on it by $\frac{1}{2m+1}$.

It now suffices to prove that

$$g(\eta^m(p)) = m!\Phi(g(p)); \tag{3.39}$$

this will show that the above expression for the right side of (3.34) is equal to the left side of (3.34). The proof of (3.39) is straightforward, using the fact that η sends the μ variables to 0. This finishes the proof of Theorem 3.26 modulo Lemma 3.38. □

Proof of Lemma 3.38. By Lemma 3.19(c) $\widetilde{\pi}_{\mathcal{A}}$ annihilates the ideal I_{δ^2} of $\widetilde{W}(\mathfrak{g})$ generated by $\operatorname{im}\delta^2$. So it suffices to prove

$$\iota_{\widehat{x}}\eta^m(p) - (\iota_x \circ g \circ \varphi)(\eta^m(p)) \in I_{\delta^2}. \tag{3.40}$$

Let $\mathrm{d}_{\widehat{K}} \in \mathrm{Der}^1\, \widetilde{W}(\mathfrak{g})$ be the Koszul differential for μ and $\widehat{\mu}$ generators defined by

$$\mathrm{d}_{\widehat{K}}(\mu) = \widehat{\mu}, \qquad \mathrm{d}_{\widehat{K}}(\widehat{\mu}) = 0. \tag{3.41}$$

Recall the derivation g of $\widetilde{W}(\mathfrak{g})$ defined by (3.32), which sends $\widehat{\mu}$ to μ and μ to 0. It is clear that the derivation $[\mathrm{d}_{\widehat{K}}, g]$ acts as multiplication by $i+j$ on $\widetilde{W}^{i,j}(\mathfrak{g})$. The map φ is the inverse of $[\mathrm{d}_{\widehat{K}}, g]$ on $\widetilde{W}^+(\mathfrak{g})$. Moreover, $g \circ \varphi$ is a homotopy operator for $\mathrm{d}_{\widehat{K}}$, i.e., if i and ε are as in Lemma 3.30, then

$$[\mathrm{d}_{\widehat{K}}, g \circ \varphi] = \mathrm{id} - i \circ \varepsilon. \tag{3.42}$$

It is clear that

$$\iota_{\widehat{x}} \circ \eta^m(p) \in \widetilde{W}^{0,+}(\mathfrak{g}), \quad \iota_x \circ g \circ \varphi \circ \eta^m(p) \in \widetilde{W}^{0,+}(\mathfrak{g}).$$

Since $\ker \mathrm{d}_{\widehat{K}} \cap \widetilde{W}^{0,+}(\mathfrak{g}) = 0$, equation (3.40) (and hence the lemma) will follow if we show that

$$(\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi)\,(\eta^m(p)) \in \ker \mathrm{d}_{\widehat{K}} + I_{\delta^2}. \tag{3.43}$$

We claim that the operator $\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi$ preserves $\ker \mathrm{d}_{\widehat{K}} + I_{\delta^2}$.

To see this, we first show that

$$[\mathrm{d}_{\widehat{K}}, \iota_{\widehat{x}}] = -\iota_x. \tag{3.44}$$

Indeed, if $\mu \in \mathfrak{g}^*$ and $x \in \mathfrak{g}$, then

$$[\mathrm{d}_{\widehat{K}}, \iota_{\widehat{x}}](\mu) = -\iota_{\widehat{x}}\, \mathrm{d}_{\widehat{K}}(\mu) = -\iota_{\widehat{x}}\widehat{\mu} = -\langle \mu, x\rangle = -\iota_x \mu,$$
$$[\mathrm{d}_{\widehat{K}}, \iota_{\widehat{x}}](\widehat{\mu}) = 0 = -\iota_x \widehat{\mu},$$

so the derivations $[\mathrm{d}_{\widehat{K}}, \iota_{\widehat{x}}]$ and $-\iota_x$ agree on generators and hence are equal. Using (3.44), (3.42), and the obvious fact $[\mathrm{d}_{\widehat{K}}, \iota_x] = 0$, we get

$$\begin{aligned}[\mathrm{d}_{\widehat{K}}, \iota_{\widehat{x}} - \iota_x \circ g \circ \varphi] &= -\iota_x + \iota_x \circ [\mathrm{d}_{\widehat{K}}, g \circ \varphi] \\ &= -\iota_x \circ \varepsilon = 0.\end{aligned}$$

So $\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi$ preserves $\ker \mathrm{d}_{\widehat{K}}$. To see that $\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi$ also preserves I_{δ^2}, we note that

$$\iota_{\widehat{x}} \circ \delta^2 = 0, \quad \iota_x \circ \delta^2 = 0, \quad g \circ \phi \circ \delta^2 = 0;$$

the first and the third equality are obvious, while the second follows from Lemma 3.16. It now follows that $\iota_{\widehat{x}}$, $g \circ \phi$ and ι_x all preserve I_{δ^2}, hence so does $\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi$. This finishes the proof of the claim that $\iota_{\widehat{x}} - \iota_x \circ g \circ \varphi$ preserves $\ker \mathrm{d}_{\widehat{K}} + I_{\delta^2}$.

We now see that (3.43) (and hence the lemma) will follow if we prove

$$\eta^m(p) \in \ker \mathrm{d}_{\widehat{K}} + I_{\delta^2} \tag{3.45}$$

For this we need the following lemma.

Lemma 3.46. *Set* $\zeta := [\eta, \mathrm{d}_{\widehat{K}}] \in \mathrm{Der}^1 \widetilde{W}(\mathfrak{g})$. *Then for* $X \in \widetilde{W}(\mathfrak{g})$ *and* $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$ *we have*

(a) $[\eta, \zeta](X) \in I_{\delta^2}$,
(b) $(\mathrm{d}_{\widetilde{W}} - \mathrm{d}_{\widehat{K}} - \zeta)(X) \in I_{\delta^2}$,
(c) $\zeta \circ \eta^k(p) \in I_{\delta^2}$.

We note that ζ is an analogue of d_{CE}.

Proof. For $\mu \in \mathfrak{g}^*$ we have

$$\zeta(\mu) = [\eta, \mathrm{d}_{\widehat{K}}](\mu) = \eta(\widehat{\mu}) - 0 = \delta(\mu),$$

which implies

$$[\eta, \zeta](\mu) = \eta \circ \delta(\mu) - \zeta(0) = 0.$$

Furthermore, we have

$$\zeta(\widehat{\mu}) = [\eta, \mathrm{d}_{\widehat{K}}](\widehat{\mu}) = 0 - \mathrm{d}_{\widehat{K}} \circ \eta(\widehat{\mu}) = -\mathrm{d}_{\widehat{K}} \circ \delta(\mu), \tag{3.47}$$

which implies

$$[\eta, \zeta](\widehat{\mu}) = \eta \circ \zeta(\widehat{\mu}) - \zeta \circ \eta(\widehat{\mu}) = -\eta \circ \mathrm{d}_{\widehat{K}} \circ \delta(\mu) - \zeta \circ \delta(\mu).$$

We rewrite this using $\eta \circ d_{\widehat{K}} = \zeta - d_{\widehat{K}} \circ \eta$ and $\eta \circ \delta(\mu) = 0$; we obtain

$$[\eta, \zeta](\widehat{\mu}) = -2\zeta \circ \delta(\mu).$$

Finally, since ζ and δ are both odd derivations of $\widetilde{W}^{0,\bullet}(\mathfrak{g})$ and since they agree on the generators μ of $\widetilde{W}^{0,\bullet}(\mathfrak{g})$, they agree on all of $\widetilde{W}^{0,\bullet}(\mathfrak{g})$, which implies $\zeta \circ \delta(\mu) = \delta^2(\mu)$. So we see that $[\eta, \zeta](\widehat{\mu}) = -2\delta^2(\mu) \in I_{\delta^2}$.

To prove (b) we see that

$$\begin{aligned}
(\mathrm{d}_{\widetilde{W}} - \mathrm{d}_{\widehat{K}})(f_i) &= \widehat{f}_i + \delta(f_i) - \widehat{f}_i = \delta(f_i) = \zeta(f_i),\\
(\mathrm{d}_{\widetilde{W}} - \mathrm{d}_{\widehat{K}})(\widehat{f}_i) &= \mathrm{d}_{\widetilde{W}}(\bar{f}_i - \delta(f_i)) = \mathrm{d}_{\widetilde{W}}(-\delta(f_i))\\
&= -\frac{1}{2}c^i_{ab}\,\mathrm{d}_{\widetilde{W}}(f_a)\otimes f_b + \frac{1}{2}c^i_{ab} f_a \otimes \mathrm{d}_{\widetilde{W}}(f_b)\\
&= -\frac{1}{2}c^i_{ab}(\widehat{f}_a + \delta(f_a))\otimes f_b + \frac{1}{2}c^i_{ab} f_a \otimes (\widehat{f}_b + \delta(f_b))\\
\text{(By (3.47))} &= \zeta(\widehat{f}_i) - \frac{1}{2}c^i_{ab}\delta(f_a)\otimes f_b + \frac{1}{2}c^i_{ab} f_a\otimes\delta(f_b)\\
&= \zeta(\widehat{f}_i) - \delta^2(f_i).
\end{aligned}$$

This proves (b).

To prove (c) we use induction on k. Assume that $k = 0$. Since $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$ we have that $\mathrm{d}_{\widetilde{W}}(p) = \mathrm{d}_{\widehat{K}}(p) = 0$. Using (b) we get that

$$\zeta(p) = (\zeta - \mathrm{d}_{\widetilde{W}} + \mathrm{d}_{\widehat{K}})(p) \in I_{\delta^2}.$$

This is the base of our induction.

Assume now that $\zeta \circ \eta^k(p) \in I_{\delta^2}$. We have that

$$\zeta \circ \eta^{k+1}(p) = \eta \circ \zeta \circ \eta^k(p) + [\zeta, \eta](\eta^k(p))$$

The first term belongs to I_{δ^2} by the induction hypothesis since η preserves I_{δ^2}. The second term belongs to I_{δ^2} by (a). Thus $\zeta \circ \eta^{k+1}(p)$ is in the ideal I_{δ^2}. □

Returning to the proof of Lemma 3.38, we first show

$$\mathrm{d}_{\widehat{K}}\, \eta^m(p) \in I_{\delta^2} \tag{3.48}$$

by using induction on k.

We note that $p \in \widetilde{\mathfrak{T}}(\mathfrak{g}) \subset \ker \mathrm{d}_{\widehat{K}}$, therefore

$$\mathrm{d}_{\widehat{K}}(\eta^0(p)) = 0,$$

thus the base case is established.

Assume now that $\mathrm{d}_{\widehat{K}}(\eta^k(p)) \in \ker \mathrm{d}_{\widehat{K}} + I_{\delta^2}$, then we have

$$\mathrm{d}_{\widehat{K}}(\eta^{k+1}(p)) = \eta(\mathrm{d}_{\widehat{K}}(\eta^k(p))) - \zeta(\eta^k(p))$$

The first term belongs to I_{δ^2} by the induction hypothesis and the fact that η preserves I_{δ^2}. The second term belongs to I_{δ^2} by 3.46(c). This proves (3.48).

Note that $i \circ \varepsilon(\eta^m(p)) = 0$ since $\eta^m(p) \in \widetilde{W}^{1,2m}(\mathfrak{g})$. Using (3.42) we get

$$\eta^m(p) = \mathrm{d}_{\widehat{K}} \circ g \circ \varphi(\eta^m(p)) + g \circ \varphi(\mathrm{d}_{\widehat{K}}(\eta^m(p))).$$

The first term belongs to $\ker \mathrm{d}_{\widehat{K}}$. Since g and φ preserve I_{δ^2}, the second term belongs to I_{δ^2}. Hence $\eta^m(p) \in \ker \mathrm{d}_{\widehat{K}} + I_{\delta^2}$, thus proving (3.45) and Lemma 3.38. □

3.4. **The quantum Weil algebra.** Suppose now $\mathfrak{g}$ admits a nondegenerate invariant symmetric bilinear form B, so we can identify $\mathfrak{g}^* \cong \mathfrak{g}$. Recall from Example 3.3 that $\mathrm{Cl}(\mathfrak{g})$ is a $\mathfrak{g}$-differential algebra with the flat connection $\vartheta_{\mathrm{Cl}(\mathfrak{g})}$ given by $\vartheta_{\mathrm{Cl}(\mathfrak{g})}(\mu) = \mu$. Let $\widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})} \colon \widetilde{W}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ be the corresponding characteristic homomorphism. For $\mu \in \mathfrak{g}^*$ we get that

$$\widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\mu) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \vartheta_{\widetilde{W}}(\mu) = \vartheta_{\mathrm{Cl}(\mathfrak{g})}(\mu) = \mu$$

and it follows from Lemma 3.19 that $\widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\widehat{\mu}) = 0$.

Define the quantum Weil algebra as

$$\mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}),$$

and let $\widehat{e}_i$ denote generators of $U(\mathfrak{g}) \otimes 1$ and e_i denote generators of $1 \otimes \mathrm{Cl}(\mathfrak{g})$. Consider the cubic Dirac element [Goe, Kos3]

$$\mathcal{D} = \sum_i^{(\mathfrak{g})} \widehat{e}_i f_i + q(\phi) \in \mathcal{W}(\mathfrak{g}),$$

where ϕ is the Cartan 3-tensor defined by (3.4) and we consider $f_i \in \mathfrak{g}^*$ as elements of $\mathfrak{g}$ via identification $\mathfrak{g}^* \cong \mathfrak{g}$ by B.

The quantum Weil algebra $\mathcal{W}(\mathfrak{g})$ is a filtered $\mathfrak{g}$-differential algebra with the differential given by $\mathrm{d}_{\mathcal{W}} = [\mathcal{D}, -]$, the Lie derivatives induced by the adjoint action and the contractions by $x \in \mathfrak{g}$ are defined by $\iota_x = \mathrm{id} \otimes \iota_x$. The filtration is induced by the usual filtrations on $U(\mathfrak{g})$ and on $\mathrm{Cl}(\mathfrak{g})$: $\deg \widehat{e}_i = 2$, $\deg e_i = 1$. We have that

$$\mathrm{d}_{\mathcal{W}}\, \mu = 2\widehat{\mu} + 2\alpha_{\mathfrak{g}}(\mu), \tag{3.49}$$

recalling $\alpha_{\mathfrak{g}}$ is defined in Equation 2.5.

Remark 3.50. Let us compare the explicit formulas for differentials in $\widetilde{W}(\mathfrak{g})$, $W(\mathfrak{g})$, and $\mathcal{W}(\mathfrak{g})$. For $\mu \in \mathfrak{g}^*$ we have

$$\begin{aligned} &\text{in } \widetilde{W}(\mathfrak{g}): && \mathrm{d}_{\widetilde{W}}(\mu) = \widehat{\mu} + \delta(\mu), \\ &\text{in } W(\mathfrak{g}): && \mathrm{d}_{W}(\mu) = 2\widehat{\mu} + 2\lambda_{\mathfrak{g}}(\mu), \\ &\text{in } \mathcal{W}(\mathfrak{g}): && \mathrm{d}_{\mathcal{W}}(\mu) = 2\widehat{\mu} + 2\alpha_{\mathfrak{g}}(\mu). \end{aligned}$$

The factor 2 that appears when we pass to $W(\mathfrak{g})$ and $\mathcal{W}(\mathfrak{g})$ is related to the Poisson structure on $W(\mathfrak{g})$, see the discussion in [Mei, §7].

One can define $\underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} : \mathcal{W}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ as the projection along $U^+(\mathfrak{g}) \otimes 1$, where $U^+(\mathfrak{g})$ is the augmentation ideal of $U(\mathfrak{g})$. This projection is a $\mathfrak{g}$-differential algebra morphism. From [Mei, Section 7.3] we have an isomorphism of $\mathfrak{g}$-differential spaces $q_W : W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ such that

$$\underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ q_W = q \circ \pi_{\wedge(\mathfrak{g})}, \tag{3.51}$$

and its restriction to $S(\mathfrak{g}) \subset W(\mathfrak{g})$, generated by $\widehat{e}_i$, is the Duflo map $S(\mathfrak{g}) \to U(\mathfrak{g})$, see [Duf]. Since q_W is an isomorphism of $\mathfrak{g}$-differential space we have that $H(\mathcal{W}(\mathfrak{g})) = \mathbb{C}$.

Using $\underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}$ we can define another transgression, $\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}$: for $x \in U^+(\mathfrak{g})^{\mathfrak{g}}$ denote by $C_x \in \mathcal{W}(\mathfrak{g})^{\mathfrak{g}}$ a cochain of transgression for x, i.e., $\mathrm{d}_{\mathcal{W}} C_x = x$. Such a cochain of transgression for x exists since there is a cochain of transgression for $q_W^{-1}(x)$. Then the transgression of x is given by $\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(x) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}(C_x)$. Note that $\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}$ depends only on the differential on $\mathcal{W}(\mathfrak{g})$ and the map $\underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}$ but not on a choice of a cochain.

Namely, suppose C_x and C'_x are two cochains of transgression for x, then $q_W^{-1}(C_x)$ and $q_W^{-1}(C'_x)$ are cochains of transgression for $y := q_W^{-1}(x) \in W(\mathfrak{g})$. Since the transgression map $\mathsf{t}_{\wedge(\mathfrak{g})}$ does not depend on a choice of a cochain (Theorem 3.6a), we have that $\pi_{\wedge\mathfrak{g}}(q_W^{-1}(C_x - C'_x)) = 0$. Thus we conclude that

$$\underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}(C_x - C'_x) = q \circ \pi_{\wedge\mathfrak{g}} \circ q_W^{-1}(C_x - C'_x) = 0,$$

which shows that $\underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{g})}$ is independent of the choice of cochain hence well-defined.

Lemma 3.52. *For $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$, we have that $q \circ \mathsf{t}_{\wedge(\mathfrak{g})}(p) = \underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{g})}(q_W(p))$.*

Proof. Since q_W intertwines the differentials we have: if C_p is a cochain of transgression for p, then $q_W(C_p)$ is a cochain of transgression for $q_W(p) \in U^+(\mathfrak{g})^{\mathfrak{g}}$. The claim now follows from (3.51). □

Corollary 3.53. *We have that $q(P_\wedge(\mathfrak{g})) = \operatorname{im} \underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{g})}$.*

Proof. Recall from Theorem 3.6(b) that $P_\wedge(\mathfrak{g})$ is equal to $\operatorname{im} \mathsf{t}_{\wedge(\mathfrak{g})}$, hence the Corollary follows from Lemma 3.52; $q \circ \mathsf{t}_{\wedge\mathfrak{g}}(p) = \underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{g})}(q_W(p))$. □

The quantum Weil algebra $\mathcal{W}(\mathfrak{g})$ also has a connection given by $\vartheta_{\mathcal{W}}(\mu) = \mu$, but this connection is not flat. By the universal property of the noncommutative Weil algebra $\widetilde{W}(\mathfrak{g})$ there exists a unique $\mathfrak{g}$-differential algebra morphism, the characteristic homomorphism, $\underline{c} : \widetilde{W}(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ such that $\underline{c} \circ \vartheta_{\widetilde{W}} = \vartheta_{\mathcal{W}}$. Before we calculate $\underline{c}$ on generators let us remark the following. Identifying $\mathfrak{g}^*$ and $\mathfrak{g}$ via bilinear form B, for $f_i \in \mathfrak{g}^* \cong \mathfrak{g}$, we have

$$\langle L_{f_i} e_a, e_b \rangle = -\langle L_{e_a} f_i, e_b \rangle = \langle f_i, [e_a, e_b] \rangle = c^i_{a,b},$$

so $L_{f_i} e_a = \sum_b c^i_{a,b} f_b$ and

$$\alpha_{\mathfrak{g}}(f_i) = \frac{1}{4} \sum_a L_{f_i} e_a \cdot f_a = \frac{1}{4} \sum_{a,b} c^i_{a,b} f_b \cdot f_a = -\frac{1}{4} \sum_{a,b} c^i_{b,a} f_b \cdot f_a. \tag{3.54}$$

Recalling $\mathrm{d}_{\mathcal{W}}(\mu) = 2\widehat{\mu} + 2\alpha_{\mathfrak{g}}(\mu)$ we get

$$\begin{aligned} \underline{c}(f_i) &= \underline{c}(\vartheta_{\widetilde{W}}(f_i)) = \vartheta_{\mathcal{W}}(f_i) = f_i, \\ \underline{c}(\widehat{f_i}) &= \underline{c}\left(\mathrm{d}_{\widetilde{W}} f_i + \frac{1}{2} \sum_{a,b} c^i_{a,b} \, f_a \otimes f_b \right) = \mathrm{d}_{\mathcal{W}} f_i + \frac{1}{2} \sum_{a,b} c^i_{a,b} \, f_a \cdot f_b \\ &= \mathrm{d}_{\mathcal{W}} f_i - 2\alpha_{\mathfrak{g}}(f_i) = 2(\widehat{f_i} + \alpha_g(f_i)) - 2\alpha_{\mathfrak{g}}(f_i) = 2\widehat{f_i}. \end{aligned}$$

Lemma 3.55. *We have that $\Phi_{\mathrm{Cl}(\mathfrak{g})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{g})\right) = \alpha_{\mathfrak{g}}(U(\mathfrak{g}))$.*

Proof. Since both $\Phi_{\mathrm{Cl}(\mathfrak{g})}$ and $\alpha_{\mathfrak{g}}$ are algebra morphisms and the map $\underline{c}$ is surjective, it suffices to show that $\Phi_{\mathrm{Cl}(\mathfrak{g})}(\widehat{\mu})$ and $\alpha_{\mathfrak{g}}(\underline{c}(\widehat{\mu}))$ are equal for $\mu \in \mathfrak{g}^*$. Take a generator f_i as before, we have

$$\Phi_{\mathrm{Cl}(\mathfrak{g})}(\widehat{f_i}) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\delta(f_i)) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}\left(-\frac{1}{2} \sum_{a,b} c^i_{a,b} \, f_a \otimes f_b \right) = -\frac{1}{2} \sum_{a,b} c^i_{a,b} \, f_a \, f_b.$$

From (3.54) we get that

$$\alpha_{\mathfrak{g}}(\underline{c}(\widehat{f}_i)) = \alpha_{\mathfrak{g}}(2\widehat{f}_i) = -\frac{1}{2}\sum_{a,b} c^i_{b,a}\, f_b\, f_a = \Phi_{\mathrm{Cl}(\mathfrak{g})}(\widehat{f}_i).$$

This proves the claim. □

Lemma 3.56. *For $p \in S^{m+1}(\mathfrak{g}^*)^{\mathfrak{g}}$, we have $\widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(\mathrm{sym}_{\widetilde{W}}(p)) = \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(\underline{c}(\mathrm{sym}_{\widetilde{W}}(p)))$.*

Proof. For $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ denote $\widetilde{p} := \mathrm{sym}_{\widetilde{W}}(p) \in \widetilde{\mathfrak{T}}(\mathfrak{g})$, then $\underline{p} := \underline{c}(\widetilde{p}) \in U^+(\mathfrak{g})^{\mathfrak{g}}$. If we denote by $C_{\widetilde{p}}$ a cochain of transgression for $\widetilde{p}$, we claim that $\underline{c}(C_{\widetilde{p}})$ is a cochain of transgression for $\underline{p}$. Indeed, since $\underline{c}$ intertwines the differentials we have

$$\mathrm{d}_{\mathcal{W}} \circ \underline{c}(C_{\widetilde{p}}) = \underline{c} \circ \mathrm{d}_{\widetilde{W}}(C_{\widetilde{p}}) = \underline{c}(\widetilde{p}) = \underline{p}.$$

Furthermore, we have $\widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})} = \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \underline{c}$; this can be checked on generators:

$$\begin{aligned} \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\mu) &= \mu, & \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \underline{c}(\mu) &= \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\mu) = \mu, \\ \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(\widehat{\mu}) &= 0, & \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \underline{c}(\widehat{\mu}) &= \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}(2\widehat{\mu}) = 0. \end{aligned}$$

Finally, we get that

$$\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(\underline{c}(\mathrm{sym}_{\widetilde{W}}(p))) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})}(C_{\underline{p}}) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \underline{c}(C_{\widetilde{p}}) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{g})}(C_{\widetilde{p}}) = \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(\mathrm{sym}_W(p)).$$

This proves the claim. □

Lemma 3.56 shows that the transgression map in $\mathcal{W}(\mathfrak{g})$ factors through $\widetilde{W}(\mathfrak{g})$ and the diagram

$$\begin{array}{ccc} S^+(\mathfrak{g}^*)^{\mathfrak{g}} & \xrightarrow{\mathrm{sym}_W} & \widetilde{\mathfrak{T}}(\mathfrak{g}) \\ \Big\downarrow{\scriptstyle q_W \circ B^\sharp} & & \Big\downarrow{\scriptstyle \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}} \\ U(\mathfrak{g})^{\mathfrak{g}} & \xrightarrow{\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}} & \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \end{array}$$

commutes. (Here $B^\sharp$ is the isomorphism of $\mathfrak{g}^*$ and $\mathfrak{g}$ induced by B.)

Theorem 3.57. *For $x \in \mathfrak{g}$, $p \in U^+(\mathfrak{g})^{\mathfrak{g}}$ we have that $\iota_x \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(p) \in \mathrm{im}\, \alpha_{\mathfrak{g}}$.*

Proof. By Theorem 7.2 on p. 174 in [Mei] the map $q_W \colon W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ factors thought the noncommutative Weil algebra:

$$W(\mathfrak{g}) \xrightarrow{\mathrm{sym}_W} \widetilde{W}(\mathfrak{g}) \xrightarrow{\underline{c}} \mathcal{W}(\mathfrak{g}).$$

Since the restriction of q_W on $S(\mathfrak{g})$ is the Duflo map, the same is true for $\underline{c} \circ \mathrm{sym}_W$. The restriction of the Dulfo map to $S(\mathfrak{g})^{\mathfrak{g}}$ defines an algebra isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$, see [Duf], the same holds for $q_W = \underline{c} \circ \mathrm{sym}_W \colon S(\mathfrak{g})^{\mathfrak{g}} \to U(\mathfrak{g})^{\mathfrak{g}}$. Therefore, it follows from Lemma 3.56 that $\mathrm{im}\, \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})} = \mathrm{im}\, \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}$. Furthermore, Theorem 3.26 shows that $\iota_x(\mathrm{im}\, \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}) \subset \Phi_{\mathrm{Cl}(\mathfrak{g})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{g})\right)$, hence

$$\iota_x(\mathrm{im}\, \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}) \subset \Phi_{\mathrm{Cl}(\mathfrak{g})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{g})\right).$$

The corollary now follows by applying the result $\Phi_{\mathrm{Cl}(\mathfrak{g})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{g})\right) = \mathrm{im}\, \alpha_{\mathfrak{g}}$ of Lemma 3.55). □

4. Transgressions in relative Weil algebras

The results of this section will allow us to extend Kostant's results [Kos2] about the structure of $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ to the relative case of $\mathrm{Cl}(\mathfrak{p})^{K}$. Analogous to the "absolute" case, the transgression maps in various relative Weil algebras all factor through the relative noncommutative Weil algebra.

4.1. **Relative noncommutative Weil algebra.** Let $(\mathfrak{g},\mathfrak{k})$ be a symmetric pair of complex Lie algebras and K be a compact Lie group such that

(a) $\mathrm{Lie}(K)=\mathfrak{k}$,
(b) the adjoint action of K on $\mathfrak{k}$ extends to an action on $\mathfrak{g}$ which preserves B, so $\mathfrak{p}$ is a K-submodule.

In particular, we are interested in the following cases $(\mathfrak{g},\mathfrak{k},K)$: $(\mathfrak{sl}(2n+1),\mathfrak{o}(2n+1),SO(2n+1))$, $(\mathfrak{sl}(2n),\mathfrak{o}(2n),O(2n))$, $(\mathfrak{sl}(2n),\mathfrak{sp}(2n),Sp(2n))$, $(\mathfrak{e}(6),\mathfrak{f}(4),F(4))$, where $F(4)$ is the compact connected simply connected group of type $\mathfrak{f}(4)$.

Let G be a compact connected Lie group and K be its symmetric subgroup, then $(\mathrm{Lie}(G),\mathrm{Lie}(K),K)$ is an example of such a triple. The only example that we consider which is not of this kind is $(\mathfrak{sl}(2n),\mathfrak{o}(2n),O(2n))$.

For such a triple $\widetilde{W}(\mathfrak{g})$ is a K-differential algebra. Define the *noncommutative relative Weil algebra* $\widetilde{W}(\mathfrak{g},K)$ to be the K-basic subalgebra of $\widetilde{W}(\mathfrak{g})$.

In what follows we denote the derivation δ in $\widetilde{W}(\mathfrak{g})$ by $\delta_{\mathfrak{g}}$. Since $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, for any $\mu\in\mathfrak{k}^*$ the element $\delta_{\mathfrak{g}}(\mu)\in\widetilde{W}^{0,2}(\mathfrak{g})$ given by (3.11) decomposes as

$$\delta_{\mathfrak{g}}(\mu)=\delta_{\mathfrak{k}}(\mu)+\delta_{\mathfrak{p}}(\mu),$$

where $\delta_{\mathfrak{k}}(\mu)\in T^2(\mathfrak{k}^*[-1])\subset\widetilde{W}^{0,2}(\mathfrak{k})\subset\widetilde{W}^{0,2}(\mathfrak{g})$ and $\delta_{\mathfrak{p}}(\mu)\in T^2(\mathfrak{p}^*[-1])\subset\widetilde{W}^{0,2}(\mathfrak{g})$. It is easy to see that this decomposition of $\delta_{\mathfrak{g}}(\mu)$ is K-equivariant. Explicitly, for the basis vector $f_i\in\mathfrak{k}^*$

$$\delta_{\mathfrak{p}}(f_i)=-\frac{1}{2}\sum_{a,b}^{(\mathfrak{p})}c^i_{a,b}f_a\otimes f_b. \tag{4.1}$$

We denote by $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}$ the restriction of linear functionals from $\mathfrak{g}$ to $\mathfrak{p}$. For $\mu\in\mathfrak{g}^*$ we set

$$\Phi_{\mathfrak{p}}(\mu)=\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\,\mu,\qquad \Phi_{\mathfrak{p}}(\widehat{\mu})=\begin{cases}\delta_{\mathfrak{p}}(\mu) & \text{if } \mu\in\mathfrak{k}^*,\\ 0 & \text{if } \mu\in\mathfrak{p}^*,\end{cases}$$

and extend it to an algebra morphism $\Phi_{\mathfrak{p}}\colon\widetilde{W}(\mathfrak{g})\to\widetilde{W}(\mathfrak{g})$. Clearly, $\Phi_{\mathfrak{p}}=\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\circ\Phi$, where the algebra morphism $\Phi\colon\widetilde{W}(\mathfrak{g})\to\widetilde{W}(\mathfrak{g})$ is defined by (3.23). Moreover, $\Phi_{\mathfrak{p}}$ is K-equivariant and preserves the $\mathfrak{k}$-horizontal subspace of $\widetilde{W}(\mathfrak{g})$, hence $\Phi_{\mathfrak{p}}$ preserves $\widetilde{W}(\mathfrak{g},K)$. In what follows we restrict $\Phi_{\mathfrak{p}}$ to an algebra morphism $\Phi_{\mathfrak{p}}\colon\widetilde{W}(\mathfrak{g},K)\to\widetilde{W}(\mathfrak{g},K)$.

Note that $\widetilde{W}(\mathfrak{g})$ is a locally free K-differential algebra with connection given by $\mu\mapsto\mu$ for $\mu\in\mathfrak{k}^*$. Let $j\colon\widetilde{W}(\mathfrak{k})\to\widetilde{W}(\mathfrak{g})$ denote the corresponding characteristic homomorphism of K-differential algebras. For $\mu\in\mathfrak{k}^*$ we have

$$\begin{aligned}&j(\mu)=\mu,\\ &j(\widehat{\mu})=j(\mathrm{d}_{\widetilde{W}}(\mu)-\delta_{\mathfrak{k}}(\mu))=\mathrm{d}_{\widetilde{W}}(\mu)-\delta_{\mathfrak{k}}(\mu)=\widehat{\mu}+\delta_{\mathfrak{g}}(\mu)-\delta_{\mathfrak{k}}(\mu)=\widehat{\mu}+\delta_{\mathfrak{p}}(\mu).\end{aligned}$$

Note that μ and $\widehat{\mu}$ denote elements of both $\widetilde{W}(\mathfrak{k})$ and $\widetilde{W}(\mathfrak{g})$; likewise $\mathrm{d}_{\widetilde{W}}$ denotes the differential of both algebras.

We identify $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$ in the usual way, and denote $\mu = \mu_\mathfrak{k} + \mu_\mathfrak{p}$ accordingly. Define the projection $\mathrm{pr}_\mathfrak{k} : \widetilde{W}(\mathfrak{g}) \to \widetilde{W}(\mathfrak{k})$ to be the algebra morphism given on the generators by $\mathrm{pr}_\mathfrak{k}(\mu) = \mu_\mathfrak{k}$, $\mathrm{pr}_\mathfrak{k}(\bar{\mu}) = \overline{\mu_\mathfrak{k}}$. Then $\mathrm{pr}_\mathfrak{k}$ is a K-differential algebra morphism which satisfies

$$\mathrm{pr}_\mathfrak{k}(\widehat{\mu}) = \mathrm{pr}_\mathfrak{k}(\widehat{\mu_\mathfrak{k}} + \widehat{\mu_\mathfrak{p}}) = \mathrm{pr}_\mathfrak{k}(\overline{\mu_\mathfrak{k}} - \delta_\mathfrak{g}(\mu_\mathfrak{k}) + \overline{\mu_\mathfrak{p}} - \delta_\mathfrak{g}(\mu_\mathfrak{p})) = \overline{\mu_\mathfrak{k}} - \delta_\mathfrak{k}(\mu_\mathfrak{k}) = \widehat{\mu_\mathfrak{k}},$$

i.e., $\mathrm{pr}_\mathfrak{k}$ takes $\widehat{\mu} \in \widetilde{W}(\mathfrak{g})$ to $\widehat{\mu_\mathfrak{k}} \in \widetilde{W}(\mathfrak{k})$.

Let $\psi_1, \psi_2 : E \to E'$ be morphisms of K-differential spaces (E, d) and (E', d'). A *K-homotopy* between ψ_1 and ψ_2 is a morphism of K-differential spaces $\psi : E \to \mathbb{C}[t, dt] \otimes E'$ such that

$$\psi_1 = (\mathrm{ev}(0) \otimes \mathrm{id}) \circ \psi, \quad \psi_2 = (\mathrm{ev}(1) \otimes \mathrm{id}) \circ \psi,$$

where $\mathrm{ev}(a)\colon \mathbb{C}[t, dt] \to \mathbb{C}$ is defined by $\mathrm{ev}(a)(P + Qdt) = P(a)$. If such a ψ exists, we say that ψ_1 and ψ_2 are *K-homotopic*. Similarly, a *K-homotopy operator* between ψ_1 and ψ_2 is a morphism of super spaces $h : E[1] \to E'$ such that

$$h \circ \iota_x + \iota_x \circ h = 0, \qquad \forall x \in \mathfrak{k},$$
$$h \circ d + d' \circ h = \psi_2 - \psi_1.$$

The maps ψ_1 and ψ_2 are K-homotopic if and only there exists a K-homotopy operator between them. We need the second part of this statement; assume there is a K-homotopy ψ between ψ_1 and ψ_2, then $h := (J \otimes \mathrm{id}) \circ \psi$ defines a K-homotopy operator between ψ_1 and ψ_2. Here J is the same "integration operator" as in 3.27.

We will say that K-differential space morphisms $\psi : E \to E'$ and $\varphi : E' \to E$ are K-homotopy inverses if $\psi \circ \varphi$ is homotopic to $\mathrm{id}_{E'}$ and $\varphi \circ \psi$ is homotopic to id_E. If a K-differential space morphism admits a K-homotopy inverse it is called a K-homotopy equivalence.

Lemma 4.2. *Let $\mathrm{pr}_\mathfrak{k} : \widetilde{W}(\mathfrak{g}) \to \widetilde{W}(\mathfrak{k})$ be as above. Then $\mathrm{pr}_\mathfrak{k}$ is a K-homotopy equivalence, with inverse j. In more detail, $\mathrm{pr}_\mathfrak{k} \circ j = \mathrm{id}_{\widetilde{W}(\mathfrak{k})}$, while a K-homotopy between $j \circ \mathrm{pr}_\mathfrak{k}$ and $\mathrm{id}_{\widetilde{W}(\mathfrak{g})}$ is given by*

$$\psi_{\mathrm{rel}} : \widetilde{W}(\mathfrak{g}) \to \mathbb{C}[t, dt] \otimes \widetilde{W}(\mathfrak{g}), \qquad \psi_{\mathrm{rel}}(\mu) = (1 - t) \otimes \mu + t \otimes j \circ \mathrm{pr}_\mathfrak{k}(\mu), \quad \mu \in \mathfrak{g}^*,$$

extended as a morphism of differential algebras using the universal property of $\widetilde{W}(\mathfrak{g})$.

Proof. Obviously we have $\mathrm{pr}_\mathfrak{k} \circ j = \mathrm{id}_{\widetilde{W}(\mathfrak{k})}$. Furthermore, it is clear that

$$\mathrm{id}_{\widetilde{W}(\mathfrak{k})} = (\mathrm{ev}(0) \otimes \mathrm{id}) \circ \psi_{\mathrm{rel}}, \quad j \circ \mathrm{pr}_\mathfrak{k} = (\mathrm{ev}(1) \otimes \mathrm{id}) \circ \psi_{\mathrm{rel}},$$

so it suffices to show that ψ_{rel} is a K-differential algebra morphism. Note first that $\mathrm{pr}_\mathfrak{k}$ is a morphism of K-differential algebras, as is j, so $j \circ \mathrm{pr}_\mathfrak{k}$ is a K-differential algebra morphism. To see that ψ_{rel} is a K-differential algebra morphism, we first describe it on generators. For this, let us introduce a differential $\mathrm{d}_\otimes$ on $\mathbb{C}[t, \mathrm{d}\, t] \otimes \widetilde{W}(\mathfrak{g})$ by $\mathrm{d}_\otimes(t) = dt$, $\mathrm{d}_\otimes(dt) = 0$, $\mathrm{d}_\otimes(\mu) = \bar{\mu}$. Now we have

$$\psi_{\mathrm{rel}}(\mu) = (1 - t) \otimes \mu + t \otimes j \circ \mathrm{pr}_\mathfrak{k}(\mu) = 1 \otimes \mu - t \otimes \mu_\mathfrak{p}$$

and

$$\begin{aligned} \psi_{\mathrm{rel}}(\bar{\mu}) &= \psi_{\mathrm{rel}}(\mathrm{d}\, \mu) = \mathrm{d}_\otimes(\psi_{\mathrm{rel}}(\mu)) = \mathrm{d}_\otimes(1 \otimes \mu - t \otimes \mu_\mathfrak{p}) \\ &= 1 \otimes \bar{\mu} - dt \otimes \mu_\mathfrak{p} - t \otimes \overline{\mu_\mathfrak{p}}. \end{aligned}$$

Since ψ_{rel} is an algebra morphism, it is enough to check the conditions on generators. Take $x \in \mathfrak{k}$ and $\mu \in \mathfrak{g}^*$ and note that

$$L_x t = L_x dt = 0, \qquad \iota_x t = \iota_x dt = 0.$$

So we get

$$\begin{aligned}\iota_x \psi_{\text{rel}}(\mu) &= \iota_x(1 \otimes \mu - t \otimes \mu_{\mathfrak{p}}) = 1 \otimes \iota_x \mu + t \otimes \iota_x \mu_{\mathfrak{p}} = 1 \otimes \iota_x \mu = \psi_{\text{rel}}(\iota_x \mu),\\ \iota_x \psi_{\text{rel}}(\bar{\mu}) &= \iota_x(1 \otimes \bar{\mu} - dt \otimes \mu_{\mathfrak{p}} - t \otimes \overline{\mu_{\mathfrak{p}}}) = 1 \otimes \iota_x \bar{\mu} - t \otimes \iota_x \overline{\mu_{\mathfrak{p}}}\\ &= 1 \otimes L_x \mu - t \otimes L_x \mu_{\mathfrak{p}} = \psi_{\text{rel}}(L_x \mu) = \psi_{\text{rel}}(\iota_x \bar{\mu}).\end{aligned}$$

It now follows from Cartan's magic formula that ψ_{rel} commutes also with Lie derivatives, which proves the claim. ☐

We denote the K-homotopy operator between $\mathrm{id}_{\widetilde{W}}$ and $j \circ \mathrm{pr}_{\mathfrak{k}}$ corresponding to ψ_{rel} by

$$h_{\text{rel}} = (J \otimes \mathrm{id}) \circ \psi_{\text{rel}}. \tag{4.3}$$

Remark 4.4. For $p \in \ker \mathrm{d}_{\widetilde{W}} \cap \widetilde{W}(\mathfrak{g}, K)$ such that $\mathrm{pr}_{\mathfrak{k}}(p) = 0$ we have

$$\mathrm{d}_{\widetilde{W}} \circ h_{\text{rel}}(p) + h_{\text{rel}} \circ \mathrm{d}_{\widetilde{W}}(p) = p - j \circ \mathrm{pr}_{\mathfrak{k}}(p), \quad \text{i.e.,} \quad \mathrm{d}_{\widetilde{W}}(h_{\text{rel}}(p)) = p.$$

Since $h_{\text{rel}}(p) \in \widetilde{W}(\mathfrak{g}, K)$, we get that $p \in \mathrm{d}_{\widetilde{W}}(\widetilde{W}(\mathfrak{g}, K))$. We use the K-homotopy operator h_{rel} below to obtain an analogue of Lemma 3.31 for the transgression map in $\widetilde{W}(\mathfrak{g}, K)$ which will be defined below.

Lemma 4.5. *For $\mu \in \mathfrak{k}^*$ we have*

$$\psi_{\text{rel}}(\widehat{\mu}) = 1 \otimes \widehat{\mu} + (2t - t^2) \otimes \delta_{\mathfrak{p}}(\mu)$$

and for $\mu \in \mathfrak{p}^*$ we have

$$\psi_{\text{rel}}(\widehat{\mu}) = (1 - t) \otimes \widehat{\mu} - dt \otimes \mu.$$

Proof. To calculate $\psi_{\text{rel}}(\widehat{\mu})$ we need $\psi_{\text{rel}}(\delta_{\mathfrak{g}}(\mu))$. Assume that our basis f_i of $\mathfrak{g}^*$ is split into $\mathfrak{k}^*$ and $\mathfrak{p}^*$, meaning that every element belongs to exactly one of them.

$$\begin{aligned}\psi_{\text{rel}}(\delta_{\mathfrak{g}}(f_i)) &= \psi_{\text{rel}}\Big(-\frac{1}{2} \sum_{a,b}^{(\mathfrak{g})} c^i_{ab} f_a \otimes f_b\Big) = -\frac{1}{2} \sum_{a,b}^{(\mathfrak{g})} c^i_{ab} \psi_{\text{rel}}(f_a) \otimes \psi_{\text{rel}}(f_b)\\ &= -\frac{1}{2} \sum_{a,b}^{(\mathfrak{g})} c^i_{a,b} \left(1 \otimes f_a - t \otimes (f_a)_{\mathfrak{p}}\right) \cdot \left(1 \otimes f_b - t \otimes (f_b)_{\mathfrak{p}}\right)\\ &= -\frac{1}{2} \sum_{a,b}^{(\mathfrak{g})} c^i_{ab} \big(1 \otimes f_a \otimes f_b - t \otimes f_a \otimes (f_b)_{\mathfrak{p}} - t \otimes (f_a)_{\mathfrak{p}} \otimes f_b + t^2 \otimes (f_a)_{\mathfrak{p}} \otimes (f_b)_{\mathfrak{p}}\big)\\ &= 1 \otimes \delta_{\mathfrak{g}}(f_i) + \frac{t}{2} \otimes \left(\sum_a^{(\mathfrak{g})} \sum_b^{(\mathfrak{p})} c^i_{ab} f_a \otimes f_b + \sum_a^{(\mathfrak{p})} \sum_b^{(\mathfrak{g})} c^i_{ab} f_a \otimes f_b \right) - \frac{t^2}{2} \otimes \sum_{a,b}^{(\mathfrak{p})} c^i_{ab} f_a \otimes f_b.\end{aligned}$$

If $f_i \in \mathfrak{k}^*$, then both sums are over basis of $\mathfrak{p}^*$ because $c^i_{ab} = 0$ if $f_a \in \mathfrak{k}^*, f_b \in \mathfrak{p}^*$ or the other way around. Also, $\sum_{a,b}^{(\mathfrak{p})} c^i_{ab} f_a \otimes f_b = -2\delta_{\mathfrak{p}}(f_i)$ so we get

$$\psi_{\text{rel}}(\delta_{\mathfrak{g}}(f_i)) = 1 \otimes \delta_{\mathfrak{g}}(f_i) - 2t \otimes \delta_{\mathfrak{p}}(f_i) + t^2 \otimes \delta_{\mathfrak{p}}(f_i).$$

On the other hand, if $f_i \in \mathfrak{p}^*$, then

$$\sum_a^{(\mathfrak{g})}\sum_b^{(\mathfrak{p})} c^i_{ab} f_a \otimes f_b + \sum_a^{(\mathfrak{p})}\sum_b^{(\mathfrak{g})} c^i_{ab} f_a \otimes f_b = \sum_a^{(\mathfrak{k})}\sum_b^{(\mathfrak{p})} c^i_{ab} f_a \otimes f_b + \sum_a^{(\mathfrak{p})}\sum_b^{(\mathfrak{k})} c^i_{ab} f_a \otimes f_b = -2\delta_{\mathfrak{g}}(f_i),$$

and the last summand in the expression above vanishes because $c^i_{ab} = 0$ if $f_i, f_a, f_b \in \mathfrak{p}^*$. (Here we are using that $\mathfrak{k}$ is symmetric in $\mathfrak{g}$.) We obtain $\psi_{\text{rel}}(\delta_{\mathfrak{g}}(f_i)) = (1-t) \otimes \delta_{\mathfrak{g}}(f_i)$.

To summarize, if $\mu \in \mathfrak{k}^*$, then

$$\psi_{\text{rel}}(\widehat{\mu}) = 1 \otimes (\bar{\mu} - \delta_{\mathfrak{g}}(\mu)) + 2t \otimes \delta_{\mathfrak{p}}(\mu) - t^2 \otimes \delta_{\mathfrak{p}}(\mu) = 1 \otimes \widehat{\mu} + (2t - t^2) \otimes \delta_{\mathfrak{p}}(\mu)$$

and if $\mu \in \mathfrak{p}^*$, then

$$\psi_{\text{rel}}(\widehat{\mu}) = 1 \otimes \bar{\mu} - dt \otimes \mu - t \otimes \bar{\mu} - (1-t) \otimes \delta_{\mathfrak{g}}(\mu) = (1-t) \otimes \widehat{\mu} - dt \otimes \mu.$$

Which concludes the proof. □

4.2. **Transgression in the noncommutative relative Weil algebra.** Let $\mathcal{A}$ be a flat locally free $\mathfrak{g}$-differential algebra such that the action of $\mathfrak{k} \subset \mathfrak{g}$ integrates to an action of K. It follows that $\mathcal{A}$ is a K-differential algebra compatible with the $\mathfrak{g}$-differential algebra structure. Set $\mathcal{B} = \mathcal{A}_{K-\text{bas}}$, the K-basic subalgebra of $\mathcal{A}$, and let $\mathcal{A}_{K-\text{hor}}$ be the K-horizontal subalgebra of $\mathcal{A}$; see Subsection 3.1. Define the algebra homomorphisms

$$\Phi^{\text{hor}}_{\mathcal{A}} := \widetilde{\pi}_{\mathcal{A}} \circ \Phi_{\mathfrak{p}} \colon \widetilde{W}(\mathfrak{g}) \to \mathcal{A}_{K-\text{hor}}. \tag{4.6}$$

It is easy to see that $\Phi_{\mathfrak{p}}(\widetilde{W}(\mathfrak{g})) \subset \widetilde{W}(\mathfrak{g})_{K-\text{hor}}$, so $\widetilde{\pi}_{\mathcal{A}} \circ \Phi_{\mathfrak{p}}(\widetilde{W}(\mathfrak{g})) \subset \mathcal{A}_{K-\text{hor}}$. Let $\widetilde{\pi}_{\mathcal{B}} \colon \widetilde{W}(\mathfrak{g}, K) \to \mathcal{B}$ be the restriction of $\widetilde{\pi}_{\mathcal{A}}$ to K-basic subalgebras. Note that $\widetilde{\pi}_{\mathcal{B}}$ is a morphism of differential algebras.

Analogously to the $\widetilde{W}(\mathfrak{g})$ case, we define the *relative transgression space* of $\widetilde{W}(\mathfrak{g}, K)$ by

$$\widetilde{\mathfrak{T}}(\mathfrak{g}, K) = \left\{ \text{sym}_W(p) \in \widetilde{W}^{+,0}(\mathfrak{g}, K) \mid p \in S^+(\mathfrak{g}^*)^K \text{ such that } \text{pr}_{\mathfrak{k}}(\text{sym}_W(p)) = 0 \right\}.$$

Since sym_W is a morphism of $\mathfrak{g}$-differential spaces, $\widetilde{\mathfrak{T}}(\mathfrak{g}, K) \subset \ker \mathrm{d}_{\widetilde{W}}$. It follows form the discussion in Remark 4.4 that for every $\widetilde{p} \in \widetilde{T}(\mathfrak{g}, K)$ there is a $\widetilde{C} \in \widetilde{W}(\mathfrak{g}, K)$ such that $\mathrm{d}_{\widetilde{W}} \widetilde{C} = \widetilde{p}$. We call such $\widetilde{C}$ a *relative cochain of transgression* for $\widetilde{p}$.

Remark 4.7. In principal, one can define a bigger transgression space as $\widetilde{W}^{+,0}(\mathfrak{g}, K) \cap \operatorname{im} \mathrm{d}_{\widetilde{W}}(\widetilde{W}(\mathfrak{g}, K))$. However, $\widetilde{\mathfrak{T}}(\mathfrak{g}, K)$ is sufficient for our purposes.

Definition 4.8. The *relative transgression map* in $\widetilde{W}(\mathfrak{g}, K)$ with values in $\mathcal{B}$ is

$$\widetilde{\mathfrak{t}}_{\mathcal{B}} \colon \widetilde{\mathfrak{T}}(\mathfrak{g}, K) \to \mathcal{B}$$

defined by sending $\widetilde{p} \in \widetilde{\mathfrak{T}}(\mathfrak{g}, K)$ to $\widetilde{\mathfrak{t}}_{\mathcal{B}}(\widetilde{p}) = \widetilde{\pi}_{\mathcal{B}}(\widetilde{C}_{\widetilde{p}})$, where $\widetilde{C}_{\widetilde{p}}$ is a relative cochain of transgression for $\widetilde{p}$.

Lemma 4.9. *The relative transgression map in $\widetilde{W}(\mathfrak{g}, K)$ is well-defined and can be expressed as $\widetilde{\mathfrak{t}}_{\mathcal{B}} = \widetilde{\pi}_{\mathcal{B}} \circ h_{\text{rel}}$.*

Proof. Let $p \in \widetilde{\mathfrak{T}}(\mathfrak{g}, K)$. Suppose C, C' are relative cochains of transgression for p. Then $\mathrm{d}_{\widetilde{W}}(C - C') = 0$, hence $(C - C') \in \ker \mathrm{d}_{\widetilde{W}}$. Thus by Lemma 3.19b $\widetilde{\pi}_{\mathcal{B}}(C - C') = \widetilde{\pi}_{\mathcal{A}}(C - C') = 0$. This proves that the relative transgression map is well-defined.

Using Remark 4.4, for $p \in \widetilde{\mathfrak{T}}(\mathfrak{g}, K)$ we get that $\mathrm{d}_{\widetilde{W}}(h_{\mathrm{rel}}(p)) = p$ and $h_{\mathrm{rel}}(p) \in \widetilde{W}(\mathfrak{g}, K)$ is a relative cochain of transgression for p. □

Proposition 4.10. *Let $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$, then*

$$\widetilde{\mathfrak{t}}_{\mathcal{B}}(p) = -\frac{4^m \cdot (m!)^2}{(2m+1)!} \Phi^{\mathrm{hor}}_{\mathcal{A}}(g(p)).$$

Explicitly, let

$$p = p' + \sum \widehat{\mu}_{i_1} \otimes \cdots \otimes \widehat{\mu}_{i_{k-1}} \otimes \widehat{\xi}_{i_k} \otimes \widehat{\mu}_{i_{k+1}} \otimes \cdots \otimes \widehat{\mu}_{i_{m+1}} \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$$

such that $\mu_{i_j} \in \mathfrak{k}^*$, $\xi_{i_j} \in \mathfrak{p}^*$ and p' does not contain any monomials of this form. Then

$$\widetilde{\mathfrak{t}}_{\mathcal{B}}(p) = -\frac{4^m \cdot (m!)^2}{(2m+1)!} \sum \Phi^{\mathrm{hor}}_{\mathcal{A}}(\widehat{\mu}_{i_1} \otimes \cdots \otimes \widehat{\mu}_{i_{k-1}}) \otimes \widetilde{\pi}_{\mathcal{A}}(\xi_{i_k}) \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\widehat{\mu}_{i_{k+1}} \otimes \cdots \otimes \widehat{\mu}_{i_{m+1}}).$$

Proof. By Lemma 4.9 and the definition (4.3) of h_{rel}, on $\widetilde{\mathfrak{T}}(\mathfrak{g}, K)$ we have

$$\widetilde{\mathfrak{t}}_{\mathcal{B}} = \widetilde{\pi}_{\mathcal{B}} \circ h_{\mathrm{rel}} = \widetilde{\pi}_{\mathcal{B}} \circ (J \otimes \mathrm{id}) \circ \psi_{\mathrm{rel}}.$$

Note that $\widetilde{\pi}_{\mathcal{A}} \circ h_{\mathrm{rel}}$ is a well-defined map on $\widetilde{W}(\mathfrak{g})$ which restricts to $\widetilde{\pi}_{\mathcal{B}} \circ h_{\mathrm{rel}}$ on $\widetilde{W}(\mathfrak{g}, K)$. Therefore we can work with monomials $\widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_{m+1} \in \widetilde{W}^{m+1,0}(\mathfrak{g})$ even though they are not in the relative transgression space $\widetilde{\mathfrak{T}}(\mathfrak{g}, K)$. Let us see which monomials $\widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_{m+1} \in \widetilde{W}^{m+1,0}(\mathfrak{g})$ will not get annihilated by the map $\widetilde{\pi}_{\mathcal{A}} \circ h_{\mathrm{rel}}$. Note that $\widetilde{\pi}_{\mathcal{A}}$ will annihilate $\widehat{\mu}$ and $J \otimes \mathrm{id}$ will send it to itself. Writing each $\psi_{\mathrm{rel}}(\widehat{\mu}_i)$ as in Lemma 4.5 we see that

(a) for $\mu_i \in \mathfrak{k}^*$ we only care about the part $(2t - t^2) \otimes \delta_{\mathfrak{p}}(\mu_i)$;
(b) for $\mu_i \in \mathfrak{p}^*$ we only care about the part $-dt \otimes \mu_i$.

Furthermore, if a monomial $\widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_{m+1}$ is such that two or more μ_is lie in $\mathfrak{p}^*$, then already ψ will annihilate it because $(dt)^2 = 0$ and if there are no such factors (meaning all $\mu_i \in \mathfrak{k}^*$) then $J \otimes \mathrm{id}$ will annihilate it, informally speaking there would be nothing to integrate over. In conclusion, the only monomials that survive the map $\widetilde{\pi}_{\mathcal{A}} \circ h_{\mathrm{rel}}$ are of the form $\widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_j \otimes \cdots \otimes \widehat{\mu}_{m+1}$, where exactly one μ_j is in $\mathfrak{p}^*$ and every other μ_i is in $\mathfrak{k}^*$. For such a monomial, recalling that $\widetilde{\pi}_{\mathcal{A}} \circ \delta_{\mathfrak{p}}(\widehat{\mu}_i) = \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_i)$, we have

$$\begin{aligned}
&\widetilde{\pi}_{\mathcal{A}} \circ h_{\mathrm{rel}}(\widehat{\mu}_1 \otimes \cdots \otimes \widehat{\mu}_j \otimes \cdots \otimes \widehat{\mu}_{m+1}) = \\
&\quad = (J \otimes \mathrm{id})\Big(\big((2t - t^2) \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_1)\big) \otimes \cdots \otimes \big(-dt \otimes \widetilde{\pi}_{\mathcal{A}}(\mu_j)\big) \otimes \cdots \\
&\qquad \cdots \otimes \big((2t - t^2) \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_{m+1})\big)\Big) \\
&\quad = -\int_0^1 (2t - t^2)^m dt \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_1) \otimes \cdots \otimes \widetilde{\pi}_{\mathcal{A}}(\mu_j) \otimes \cdots \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_{m+1}) \\
&\quad = -\frac{4^m \cdot (m!)^2}{(2m+1)!} \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_1) \otimes \cdots \otimes \widetilde{\pi}_{\mathcal{A}}(\mu_j) \otimes \cdots \otimes \Phi^{\mathrm{hor}}_{\mathcal{A}}(\mu_{m+1}).
\end{aligned}$$

Now note that $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$ can be written as

$$p = p' + \sum \widehat{\mu}_{i_1} \otimes \cdots \otimes \widehat{\mu}_{i_{k-1}} \otimes \widehat{\xi}_{i_k} \otimes \widehat{\mu}_{i_{k+1}} \otimes \cdots \otimes \widehat{\mu}_{i_{m+1}}$$

so that $\mu_{i_j} \in \mathfrak{k}^*$ and $\xi_{i_j} \in \mathfrak{p}^*$ and p' has no summands of this form, meaning that each summand in p' contains at least 2 factors $\widehat{\xi}$, for $\xi \in \mathfrak{p}^*$, so $\widetilde{\pi}_{\mathcal{B}} \circ h_{\mathrm{rel}}(p') = 0$. □

Corollary 4.11. *For all* $\widetilde{p} \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$ *we have that*

$$\widetilde{\mathfrak{t}}_{(\bigwedge \mathfrak{p}^*)^K}(\widetilde{p}) = -4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \widetilde{\mathfrak{t}}_{\bigwedge \mathfrak{g}^*}(\widetilde{p}), \qquad \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\widetilde{p}) = -4^m q \circ \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ q^{-1} \circ \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{g})}(\widetilde{p}).$$

Proof. For $\mathcal{A} = \bigwedge \mathfrak{g}$ or $\mathcal{A} = \mathrm{Cl}(\mathfrak{g})$ and $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K) \subseteq \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g})$, using Proposition 3.33, we have

$$\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \widetilde{\mathfrak{t}}_{\mathcal{A}}(\widetilde{p}) = \frac{(m!)^2}{(2m+1)!} \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \Phi_{\mathcal{A}} \circ g(\widetilde{p})$$

(Since $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \widetilde{\pi}_{\mathcal{A}} = \widetilde{\pi}_{\mathcal{A}} \circ \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}}$)

$$= \frac{(m!)^2}{(2m+1)!} \widetilde{\pi}_{\mathcal{A}} \circ \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \Phi \circ g(\widetilde{p})$$

(Using that $\Phi^{\mathrm{hor}}_{\mathcal{A}} = \widetilde{\pi}_{\mathcal{A}} \circ \Phi_{\mathfrak{p}} = \widetilde{\pi}_{\mathcal{A}} \circ \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \Phi$.)

$$= \frac{(m!)^2}{(2m+1)!} \Phi^{\mathrm{hor}}_{\mathcal{A}} \circ g(\widetilde{p})$$

(Using Proposition 4.10.)

$$= -\frac{1}{4^m} \widetilde{\mathfrak{t}}_{\mathcal{B}}(\widetilde{p})$$

Which proves the claim. □

Proposition 4.12. *Let* $\mathcal{A} = \bigwedge \mathfrak{g}^*$ *or* $\mathcal{A} = \mathrm{Cl}(\mathfrak{g})$, *so* $\mathcal{B} = (\bigwedge \mathfrak{p}^*)^K$, *respectively* $\mathcal{B} = \mathrm{Cl}(\mathfrak{p})^K$. *For* $x \in \mathfrak{p}$ *and* $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$ *we have*

$$\iota_x \widetilde{\mathfrak{t}}_{\mathcal{B}}(p) = -\frac{4^m (m!)^2}{(2m)!} \Phi^{\mathrm{hor}}_{\mathcal{A}}(\iota_{\widehat{x}} p)$$

Proof. By Theorem 3.26, for $p \in \widetilde{\mathfrak{T}}^{m+1}(\mathfrak{g}, K)$ and $x \in \mathfrak{p}$ we have

$$\iota_x \widetilde{\mathfrak{t}}_{\mathcal{A}}(p) = \frac{(m!)^2}{(2m)!} \Phi_{\mathcal{A}}(\iota_{\widehat{x}} p).$$

Now, using Corollary 4.11 and noting that the maps ι_x, $x \in \mathfrak{p}$ commute with $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}}$ on $\mathcal{A}$, we have

$$\begin{aligned} \iota_x \widetilde{\mathfrak{t}}_{\mathcal{B}}(p) &= \iota_x \left(-4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \widetilde{\mathfrak{t}}_{\mathcal{A}}(p)\right) = -4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \iota_x \circ \widetilde{\mathfrak{t}}_{\mathcal{A}}(p) \\ &= -4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \left(\frac{(m!)^2}{(2m)!} \Phi_{\mathcal{A}}(\iota_{\widehat{x}} p) \right) = -\frac{4^m (m!)^2}{(2m)!} \Phi^{\mathrm{hor}}_{\mathcal{A}}(\iota_{\widehat{x}} p). \end{aligned}$$

Which proves the claim. □

4.3. **Transgression in the commutative relative Weil algebra.** Define the relative commutative Weil algebra

$$W(\mathfrak{g}, K) = W(\mathfrak{g})_{K-bas} = \left(S(\mathfrak{g}^*) \otimes \bigwedge \mathfrak{p}^*\right)^K.$$

The *relative transgression space* in $W(\mathfrak{g}, K)$ is

$$\mathfrak{T}(\mathfrak{g}, K) = \left\{ p \in W^{+,0}(\mathfrak{g}, K) \mid \exists C_p \in \widetilde{W}(\mathfrak{g}, K) : p = \mathrm{d}_W C_p \right\}.$$

For $p \in \mathfrak{T}(\mathfrak{g}, K)$ an element C_p as above is called a *relative cochain of transgression* for p.

Let $\mathcal{A}$ be a commutative $\mathfrak{g}$-differential algebra with a flat connection and set $\mathcal{B} = \mathcal{A}_{K-\mathrm{bas}}$ (below we will specialise to the case $\mathcal{A} = \bigwedge\mathfrak{g}^*$). Then the relative transgression map $\mathsf{t}_{\mathcal{B}} \colon \mathfrak{T}(\mathfrak{g}, K) \to \mathcal{B}$ for commutative Weil algebra (also considered in [Gri]) is defined as follows.

Definition 4.13. For $p \in \mathfrak{T}(\mathfrak{g}, K)$ let $C \in W(\mathfrak{g}, K)$ be a relative cochain of transgression for $= p$. The linear map $\mathsf{t}_{\mathcal{B}} \colon \mathfrak{T}(\mathfrak{g}, K) \to \mathcal{B}$ defined by $\mathsf{t}_{\mathcal{B}}(p) = \pi_{\mathcal{B}}(C)$ is called the *relative transgression map* in $W(\mathfrak{g}, K)$ with values in $\mathcal{B}$.

Similar arguments to the proof of Lemma 4.9 show that the relative transgression map $\mathsf{t}_{\mathcal{B}}$ in $W(\mathfrak{g}, K)$ does not depend on the choice of relative cochain of transgression and, hence, is well-defined.

Lemma 4.14. *For $p \in T(\mathfrak{g}, K)$ we have that*

$$\mathsf{t}_{(\wedge\mathfrak{p}^*)^K}(p) = -4^m \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathsf{t}_{\wedge\mathfrak{g}^*}(p)$$

Proof. Let $C_p \in W(\mathfrak{g})$ be a cochain of transgression for p, then $\mathrm{sym}_W(C_p) \in \widetilde{W}(\mathfrak{g})$ is a cochain of transgression for $\mathrm{sym}_W(p)$.

$$\mathsf{t}_{\wedge\mathfrak{g}^*}(p) = \widetilde{\mathsf{t}}_{\wedge\mathfrak{g}^*}(\mathrm{sym}(p)).$$

In a similar manner, one can show that

$$\mathsf{t}_{(\wedge\mathfrak{p}^*)^K}(p) = \widetilde{\mathsf{t}}_{(\wedge\mathfrak{p}^*)^K}(\mathrm{sym}(p)).$$

The claim of the Lemma now follows from Corollary 4.11. □

4.4. **Transgression in the relative quantum Weil algebra.** Define the relative quantum Weil algebra by

$$\mathcal{W}(\mathfrak{g}, K) = \mathcal{W}(\mathfrak{g})_{K-\mathrm{bas}} = (U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}))^K.$$

From [Mei, §7.4] we have that $H(W(\mathfrak{g}, K)) \simeq S(\mathfrak{k})^K$ and $H(\mathcal{W}(\mathfrak{g}, K)) \simeq S(\mathfrak{k})^K$.

The *relative transgression space* in $\mathcal{W}(\mathfrak{g}, K)$ by

$$\underline{\mathfrak{T}}(\mathfrak{g}, K) = \left\{ \underline{p} \in \mathcal{W}^{+,0}(\mathfrak{g}, K) \mid \exists \underline{C}_{\underline{p}} \in \widetilde{W}(\mathfrak{g}, K) : \underline{p} = \mathrm{d}_{\mathcal{W}} \underline{C}_{\underline{p}} \right\}.$$

For $\underline{p} \in \underline{\mathfrak{T}}(\mathfrak{g}, K)$ an element $\underline{C}_{\underline{p}}$ as above is called a *relative cochain of transgression* for $\underline{p}$.

Note that since $q_W \colon W(\mathfrak{g}) \to \mathcal{W}(\mathfrak{g})$ is a morphism of $\mathfrak{g}$-differential spaces, it can be restricted to K-basic subalgebras. The restriction $q_W \colon W(\mathfrak{g}, K) \to \mathcal{W}(\mathfrak{g}, K)$ is a morphism of differential spaces. Therefore, we have that

$$\underline{\mathfrak{T}}(\mathfrak{g}, K) = q_W(\mathfrak{T}(\mathfrak{g}, K)). \tag{4.15}$$

Definition 4.16. For $\underline{p} \in \underline{\mathfrak{T}}(\mathfrak{g}, K)$ let $\underline{C} \in \mathcal{W}(\mathfrak{g}, K)$ be a corresponding relative cochain of transgression such that $\mathrm{d}_{\mathcal{W}} \underline{C} = \underline{p}$. The linear map $\underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{p}) \colon \underline{\mathfrak{T}}(\mathfrak{g}, K) \to \mathrm{Cl}(\mathfrak{p})^K$ defined by $\underline{\mathsf{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{p}) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{C}_{\underline{p}})$ is called the *relative transgression map* in $\mathcal{W}(\mathfrak{g}, K)$.

The relative transgression map in $\mathcal{W}(\mathfrak{g}, K)$ is well-defined by similar arguments as in Subsection 3.4.

Denote the algebra morphism $\Phi^{\mathrm{hor}}_{\mathrm{Cl}(\mathfrak{g})} \colon \widetilde{W}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{p})$ which is defined in (4.6) by $\Phi_{\mathrm{Cl}(\mathfrak{p})}$.

Lemma 4.17. *We have* $\Phi_{\mathrm{Cl}(\mathfrak{p})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{k})\right) = \alpha_{\mathfrak{p}}(U(\mathfrak{k}))$.

Proof. By abuse of notation let $\underline{c}$ be the characteristic homomorphism $\widetilde{W}(\mathfrak{k}) \to \mathcal{W}(\mathfrak{k})$. Since both $\Phi_{\mathrm{Cl}(\mathfrak{p})}$ and $\alpha_{\mathfrak{p}}$ are linear and the map $\underline{c}$ is surjective, it suffices to show that $\Phi_{\mathrm{Cl}(\mathfrak{p})}(\widehat{\mu})$ and $\alpha_{\mathfrak{p}}(\underline{c}(\widehat{\mu}))$ are equal for $\mu \in \mathfrak{k}^*$. Take a generator $\widehat{f_i}$ for $f_i \in \mathfrak{k}^*$, we have

$$\Phi_{\mathrm{Cl}(\mathfrak{p})}(\widehat{f_i}) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{p})}(\delta_{\mathfrak{p}}(f_i)) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{p})}\left(-\frac{1}{2}\sum_{a,b}^{(\mathfrak{p})} c^i_{a,b}\, f_a \otimes f_b\right) = -\frac{1}{2}\sum_{a,b}^{(\mathfrak{p})} c^i_{a,b}\, f_a\, f_b,$$

where we used the formula (4.1) for $\delta_{\mathfrak{p}}$. From (3.54) we get that

$$\alpha_{\mathfrak{p}}(\underline{c}(\widehat{f_i})) = \alpha_{\mathfrak{p}}(2\widehat{f_i}) = -\frac{1}{2}\sum_{a,b}^{(\mathfrak{p})} c^i_{b,a}\, f_b\, f_a = \Phi_{\mathrm{Cl}(\mathfrak{p})}(\widehat{f_i}).$$

This proves the claim. □

Lemma 4.18. *For any* $p \in \mathfrak{T}^{m+1}(\mathfrak{g}, K)$*,* $q \circ \mathfrak{t}_{(\wedge\mathfrak{p})^K}(p) = \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(q_W(p))$*.*

Proof. Let $C_p \in W(\mathfrak{g}, K)$ be a relative cochain of transgression for p. Then by (4.15), $q_W(C_p) \in \mathcal{W}(\mathfrak{g}, K)$ is a relative cochain of transgression for $q_W(p)$. Since the restriction of q_W to $\bigwedge\mathfrak{p}$ is q we have that

$$q \circ \pi_{(\wedge\mathfrak{p})^K}(C_p) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{p})^K}(q_W(C_p)).$$

Which proves the claim. □

Lemma 4.19. *For* $p \in T^{m+1}(\mathfrak{g}, K)$*,*

$$\widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\mathrm{sym}_{\widetilde{W}}(p)) = \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{c}(\mathrm{sym}_{\widetilde{W}}(p))).$$

Proof. First note that it is easy to check that $\widehat{\pi}_{\mathrm{Cl}(\mathfrak{g})} = \underline{\pi}_{\mathrm{Cl}(\mathfrak{g})} \circ \underline{c}$. Hence $\widehat{\pi}_{\mathrm{Cl}(\mathfrak{p})^K} = \underline{\pi}_{\mathrm{Cl}(\mathfrak{p})^K} \circ \underline{c}$. Let $\widetilde{C}_p$ be a relative cochain of transgression for $\mathrm{sym}_{\widetilde{W}}(p)$, then $\underline{c}(\widetilde{C}_p)$ is a relative cochain of transgression for $\underline{c}(\mathrm{sym}_{\widetilde{W}}(p))$. We have

$$\widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\mathrm{sym}_{\widetilde{W}}(p)) = \widetilde{\pi}_{\mathrm{Cl}(\mathfrak{p})^K}(\widetilde{C}_p) = \underline{\pi}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{c}(\widetilde{C}_p)) = \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(\underline{c}(\mathrm{sym}_{\widetilde{W}}(p))),$$

which proves the claim. □

Theorem 4.20. *For any* $x \in \mathfrak{p}$ *and* $p \in \underline{\mathfrak{T}}(\mathfrak{g}, K)$*,* $\iota_x \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}(p) \in \mathrm{im}\, \alpha_{\mathfrak{p}}$*.*

Proof. Using the same argument as in Theorem 3.57 we can conclude $\mathrm{im}\, \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K} = \mathrm{im}\, \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}$ from Lemma 4.19. From Proposition 4.12 we see that $\iota_x(\mathrm{im}\, \widetilde{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}) \subset \Phi_{\mathrm{Cl}(\mathfrak{p})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{k})\right)$, so $\iota_x(\mathrm{im}\, \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})}) \subset \Phi_{\mathrm{Cl}(\mathfrak{p})}\left(\widetilde{W}^{\bullet,0}(\mathfrak{k})\right)$, which is equal to $\mathrm{im}\, \alpha_{\mathfrak{p}}$ by Lemma 4.17. □

This theorem also shows that the image of the transgression map $\underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}$ consists of primitive elements in the sense of [Ras, §5]. This fact motivates our definition of the primitive invariants in $\mathrm{Cl}(\mathfrak{p})$, see Section 5.

5. Primitive invariants and relative transgression theorem for primary and almost primary types

In this section we define the space of primitive K-invariants in $\bigwedge\mathfrak{p}^*$ and prove the relative analogue of the celebrated Transgression Theorem 3.6.

We specify $(\mathfrak{g}, \mathfrak{k}, K)$ to be in the following list:

$$\begin{array}{ll} \text{(a)} & (\mathfrak{sl}(2n+1), \mathfrak{o}(2n+1), SO(2n+1)),\\ \text{(b)} & (\mathfrak{sl}(2n), \mathfrak{o}(2n), O(2n)),\\ \text{(c)} & (\mathfrak{sl}(2n), \mathfrak{sp}(2n), Sp(2n)),\\ \text{(d)} & (\mathfrak{e}(6), \mathfrak{f}(4), F(4)). \end{array} \tag{5.1}$$

Lemma 5.2. *For $(\mathfrak{g}, \mathfrak{k}, K)$ as in the list (5.1) we have that*

$$\mathrm{pr}_{\mathfrak{k}} \colon S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^K$$

is surjective and admits a splitting map

$$s \colon S(\mathfrak{k}^*)^K \to S(\mathfrak{g}^*)^{\mathfrak{g}}$$

which is an algebra map.

Proof. We first tackle items (a),(c),(d) on our list (5.1), noting that in these settings K is connected and $S(\mathfrak{k}^*)^K = S(\mathfrak{k}^*)^{\mathfrak{k}}$. We have the following commutative diagram of Harish-Chandra isomorphisms

$$\begin{array}{ccc} S(\mathfrak{g}^*)^{\mathfrak{g}} & \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} & S(\mathfrak{k}^*)^{\mathfrak{k}} \\ \Big\downarrow{\scriptstyle \mathrm{hc}} & & \Big\downarrow{\scriptstyle \mathrm{hc}} \\ S(\mathfrak{h}^*)^{W_{\mathfrak{g}}} & \xrightarrow{\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}}} & S(\mathfrak{k}^*)^{W_{\mathfrak{k}}}. \\ \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\ S(P(W_{\mathfrak{g}})) & \xrightarrow{\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}}} & S(P(W_{\mathfrak{k}})) \end{array}$$

where $P(W_{\mathfrak{g}})$ and $P(W_{\mathfrak{k}})$ are primitive invariant polynomials in $\mathfrak{h}^*$ and $\mathfrak{t}^*$ respectively. In (a) (resp. (c)), let $\mathfrak{h}$ be traceless diagonal $(2n+1) \times (2n+1)$ (resp $2n \times 2n$) matrices, and let $x_i : \mathfrak{h} \to \mathbb{C}$ be the functional that picks out the ith entry. The space $P(W_{\mathfrak{g}})$ can be taken to be the span of the degree 2 to $2n+1$ (or $2n$) power sum symmetric polynomials in x_i. Let $\mathfrak{t}$ be diagonal matrices which are antisymmetric under reflection along the antidiagonal, let $\{y_i \in \mathfrak{t}^* : i \in 1, \cdots, n\}$ pick out the ith entry. Then $P(W_{\mathfrak{k}})$ could be the span of the first n power sum polynomials in y_i^2. For $i = 1, \cdots, n$, $\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}}(x_i) = y_i$ and $\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}}(x_{n+i}) = -y_i$, hence the restriction to $\mathfrak{t}$ of an odd power sum polynomial in x_i is zero and the restriction of an even power sum polynomial is twice the power sum polynomial in $\{y_i^2\}$. Thus $\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}}$ is surjective and we construct a splitting by lifting the degree j power sum polynomial in y_i^2 to degree $2j$ power sum polynomial in x_i for $j = 1, \ldots, n$.

For (d) one could check by computer or appeal to [Oni, Theorem 1 on p. 216] which states that when $(\mathfrak{g}, \mathfrak{k})$ is primary then $\mathrm{pr}_{\mathfrak{k}} : S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{k}^*)^{\mathfrak{k}}$ is surjective and a splitting can be constructed using any preimage of the generators for $S(\mathfrak{h}^*)^{\mathfrak{k}}$, this will be an algebra morphism since $S(\mathfrak{h}^*)^{\mathfrak{k}}$ is a free commutative algebra.

When the triple $(\mathfrak{g}, \mathfrak{k}, K)$ is in the category (b) then $P(W_{\mathfrak{g}})$ can again be taken to be the first $2n$ power sum polynomials in x_i, and again $\mathfrak{t}$ are diagonal matrices which are antisymmetric under reflection along the antidiagonal. Note that $P(W_{\mathfrak{k}})$ can be the first $n-1$ power sum polynomials in y_i^2 and the Pfaffian $\prod_{i=1}^n y_i$. In this case $\mathrm{pr}_{\mathfrak{k}}$ does not surject onto $S(\mathfrak{h}^*)^{\mathfrak{k}}$. However, the Pfaffian is not K invariant; the algebra $S(\mathfrak{k}^*)^K$ is congruent under the Harish-Chandra isomorphism to $S(\mathfrak{t}^*)^{W_K}$, with $W_K \cong W_{B_n}$ (see [CNP, §3.8]). This algebra is equal to the polynomial algebra (of rank $\dim \mathfrak{t}$) generated by the first n power sum polynomials in y_i^2. Using the reasoning for (c), this is clearly the image of $\mathrm{Res}^{\mathfrak{h}}_{\mathfrak{t}} : S(\mathfrak{h}^*)^{W_{\mathfrak{g}}} \to S(\mathfrak{t}^*)^{W_{\mathfrak{t}}}$. Again a splitting lifts the degree j symmetric power sum polynomials in y_i^2 to degree $2j$ symmetric power sum polynomials in x_i. □

Lemma 5.3. *Consider* $(\mathfrak{g}, \mathfrak{k}, K)$ *as in the list* (5.1)*, then the sequence*

$$0 \longrightarrow \mathfrak{T}(\mathfrak{g}, K) \hookrightarrow S^+(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} S^+(\mathfrak{k}^*)^K \longrightarrow 0$$

is exact and split.

Proof. Note that for any $x \in \mathfrak{g}$ and $p \in \mathfrak{T}(\mathfrak{g}, K)$, since $\mathrm{d}_W\, p = 0$ and $p \in S^+(\mathfrak{g}^*)^K$ we have

$$L_x p = \mathrm{d}_W \circ \iota_x(p) + \iota_x \circ \mathrm{d}_W(p) = 0.$$

Hence, $p \in W^{+,0}(\mathfrak{g})^{\mathfrak{g}}$. Thus $\mathfrak{T}(\mathfrak{g}, K) \subseteq \mathfrak{T}(\mathfrak{g}) = S^+(\mathfrak{g}^*)^{\mathfrak{g}}$.

By abuse of notation, let $j\colon W(\mathfrak{k}) \to W(\mathfrak{g})$ denote the characteristic homomorphism and $\mathrm{pr}_{\mathfrak{k}}\colon W(\mathfrak{g}) \to W(\mathfrak{k})$ is defined by the same formulas on generators of $W(\mathfrak{g})$ as in Lemma 4.2. Similarly to §4.1, we can show that there is a K-homotopy operator $h_{\mathrm{rel}}\colon W(\mathfrak{g}) \to W(\mathfrak{g})$ such that

$$h_{\mathrm{rel}} \circ \mathrm{d}_W + \mathrm{d}_W \circ h_{\mathrm{rel}} = \mathrm{id} - j \circ \mathrm{pr}_{\mathfrak{k}}\,. \tag{5.4}$$

Suppose that $p \in \mathfrak{T}(\mathfrak{g}, K)$, therefore, $[p] \in H(\mathfrak{g}, K)$ is zero. Since $\mathrm{pr}_{\mathfrak{k}}$ induces an isomorphism in cohomology, $[\mathrm{pr}_{\mathfrak{k}}(p)] \in H(S(\mathfrak{k}^*)^K)$ is zero too. The differential on $S(\mathfrak{k}^*)^K$ is zero, thus, $[\mathrm{pr}_{\mathfrak{k}}(p)] = \mathrm{pr}_{\mathfrak{k}}(p) = 0$. Therefore, $\mathfrak{T}(\mathfrak{g}, K) \subseteq \ker \mathrm{pr}_{\mathfrak{k}} \cap S^+(\mathfrak{g}^*)^{\mathfrak{g}}$.

Assume that $p \in S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cap \ker \mathrm{pr}_{\mathfrak{k}}$. Applying (5.4) to p,

$$\mathrm{d}_W \circ h_{\mathrm{rel}}(p) = p.$$

Since h_{rel} is a K-homotopy operator, by passing to K-basic subspaces we get a homotopy equivalence between $j\colon S(\mathfrak{k}^*)^K \to W(\mathfrak{g}, K)$ and $\mathrm{pr}_{\mathfrak{k}}\colon W(\mathfrak{g}, K) \to S(\mathfrak{k}^*)^K$ given by $h_{\mathrm{rel}}\colon W(\mathfrak{g}, K) \to W(\mathfrak{g}, K)$. Therefore, $h_{\mathrm{rel}}(p) \in W(\mathfrak{g}, K)$ and hence it is a relative cochain of transgression for p. So $\ker \mathrm{pr}_{\mathfrak{k}} \cap S^+(\mathfrak{g}^*)^{\mathfrak{g}} \subseteq \mathfrak{T}(\mathfrak{g}, K)$.

It was shown in Lemma 5.2 that the map $\mathrm{pr}_{\mathfrak{k}}\colon S^+(\mathfrak{g}^*)^{\mathfrak{g}} \to S^+(\mathfrak{k}^*)^K$ is surjective and admits a splitting map. □

Definition 5.5. The $\mathbb{Z}$-graded subspace $P_\wedge(\mathfrak{p}) := \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} P_\wedge(\mathfrak{g}) \subset (\bigwedge \mathfrak{p}^*)^{\mathfrak{k}}$ is called the space of *primitive invariants* in $\bigwedge \mathfrak{p}^*$.

From the explicit form of primitives $P_\wedge(\mathfrak{p})$ for $(\mathfrak{sl}(2n), \mathfrak{so}(2n))$ given in [Dol2], it is easy to see that the primitives are in fact not only $\mathfrak{k}$-invariant but also K-invariant. (They are given as traces, hence are invariant under conjugation by K.)

It follows from [HS, Theorem 12], see also the discussion in [Dol1, §4.2] and [Dol2], that for triples $(\mathfrak{g}, \mathfrak{k}, K)$ from the list (5.1) we have

$$\dim P_\wedge(\mathfrak{p}) = \mathrm{rank}\, \mathfrak{g} - \mathrm{rank}\, \mathfrak{k} = \dim \mathfrak{a}. \tag{5.6}$$

We claim that the space $\mathfrak{T}(\mathfrak{g}, K)$ admits a structure of a $S(\mathfrak{g}^*)^{\mathfrak{g}}$-module induced by the multiplication in the symmetric algebra $S(\mathfrak{g}^*)$. Namely, let $p \in \mathfrak{T}(\mathfrak{g}, K)$ and $x \in S(\mathfrak{g}^*)^{\mathfrak{g}}$. Let $C_p \in W(\mathfrak{g}, K)$ be a relative cochain of transgression for p. Then

$\mathrm{d}_W(xC_p) = xp$. Since $xC_p \in W(\mathfrak{g}, K)$, xC_p is a relative cochain of transgression for xp. Clearly, if $x \in S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ we have that

$$\mathsf{t}_{\mathcal{B}}(xp) = \pi_{\mathcal{A}}(xC_p) = \pi_{\mathcal{A}}(x)\pi_{\mathcal{A}}(C_p) = 0 \tag{5.7}$$

since $\pi_{\mathcal{A}}(x)$ vanishes by Lemma 3.19a. It also follows that $\mathfrak{T}(\mathfrak{g}, K)$ is an ideal in $S^+(\mathfrak{g}^*)^{\mathfrak{g}}$.

The following theorem is an extended version of [Gri, Theorem 4.3.11].

Theorem 5.8 (Relative transgression theorem)**.** *Let $(\mathfrak{g}, \mathfrak{k}, K)$ be as in the list* (5.1), *then the transgression map $\mathsf{t}_{(\wedge\mathfrak{p}^*)^K}$ satisfies*

$$\ker \mathsf{t}_{(\wedge\mathfrak{p}^*)^K} = S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cdot \mathfrak{T}(\mathfrak{g}, K), \qquad \operatorname{im} \mathsf{t}_{(\wedge\mathfrak{p}^*)^K} = P_\wedge(\mathfrak{p}).$$

In particular, the map

$$\mathsf{t}_{(\wedge\mathfrak{p}^*)^K} \colon \mathfrak{T}(\mathfrak{g}, K) \big/ S^+(\mathfrak{g}^*)^{\mathfrak{g}} \cdot \mathfrak{T}(\mathfrak{g}, K) \to P_\wedge(\mathfrak{p}). \tag{5.9}$$

is an isomorphism of linear spaces.

Proof. By (5.7), $S^+(\mathfrak{g})^{\mathfrak{g}} \cdot \mathfrak{T}(\mathfrak{g}, K) \subseteq \ker \mathsf{t}_{(\wedge\mathfrak{p}^*)^K}$.

By Lemma 5.3 the short exact sequence

$$0 \longrightarrow \mathfrak{T}(\mathfrak{g}, K) \hookrightarrow S^+(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} S^+(\mathfrak{k}^*)^K \longrightarrow 0 \tag{5.10}$$

is split. Let $s\colon S^+(\mathfrak{k}^*)^K \to S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ be the splitting map constructed in Lemma 5.2. We have

$$S^+(\mathfrak{g}^*)^{\mathfrak{g}} = \mathfrak{T}(\mathfrak{g}, K) \oplus s(S^+(\mathfrak{k}^*)^K) \cong \mathfrak{T}(\mathfrak{g}, K) \oplus S^+(\mathfrak{k}^*)^K. \tag{5.11}$$

In particular, noting that $\mathfrak{T}(\mathfrak{g}, K) \cdot S^+(\mathfrak{g}^*)^{\mathfrak{g}}$ is an ideal in $\mathfrak{T}(\mathfrak{g}, K)$ and that $\mathfrak{T}(\mathfrak{g}, K)$ is an ideal in $S^+(\mathfrak{g}^*)^{\mathfrak{g}}$, we have that

$$\big(S^+(\mathfrak{g}^*)^{\mathfrak{g}}\big)^2 \cong \mathfrak{T}(\mathfrak{g}, K) \cdot S^+(\mathfrak{g}^*)^{\mathfrak{g}} \oplus \big(S^+(\mathfrak{k}^*)^K\big)^2. \tag{5.12}$$

Since (by Chevalley's famous result) $S(\mathfrak{g}^*)^{\mathfrak{g}} \cong S(\mathfrak{h}^*)^{W_{\mathfrak{g}}}$ is a polynomial algebra of rank $\dim \mathfrak{h}$ and $S(\mathfrak{k}^*)^K \cong S(\mathfrak{t}^*)^{W_K}$ is a polynomial algebra of rank $\dim \mathfrak{t}$, it follows that

$$\dim S^+(\mathfrak{g})^{\mathfrak{g}} \big/ \big(S^+(\mathfrak{g})^{\mathfrak{g}}\big)^2 = \dim \mathfrak{h} \quad \text{and} \quad \dim S^+(\mathfrak{k})^K \big/ \big(S^+(\mathfrak{k})^K\big)^2 = \dim \mathfrak{t}.$$

Therefore, using (5.11) and (5.12), we see that

$$\dim \mathfrak{T}(\mathfrak{g}, K) \big/ S^+(\mathfrak{g})^{\mathfrak{g}} \cdot \mathfrak{T}(\mathfrak{g}, K) = \dim \mathfrak{h} - \dim \mathfrak{t} = \dim \mathfrak{a}.$$

Let $P(W_{\mathfrak{g}})$ and $P(W_K)$ be as in the proof of Lemma 5.2. Set

$$P_S(\mathfrak{g}) = \mathrm{hc}^{-1}(P(W_{\mathfrak{g}})) \cong S^+(\mathfrak{g})^{\mathfrak{g}} \big/ \big(S^+(\mathfrak{g})^{\mathfrak{g}}\big)^2,$$

$$P_S(K) = \mathrm{hc}^{-1}(P(W_K)) \cong S^+(\mathfrak{k})^K \big/ \big(S^+(\mathfrak{k})^K\big)^2.$$

It follows from Lemma 5.2 that the image of $P_S(\mathfrak{g})$ under $\mathrm{pr}_{\mathfrak{k}}$ is $P_S(K)$ and the splitting map s sends $P_S(K)$ to $P_S(\mathfrak{g})$.

Furthermore, let

$$P_S(\mathfrak{g}, \mathfrak{k}) := P_S(\mathfrak{g}) \cap \mathfrak{T}(\mathfrak{g}, K).$$

Using the split exact sequence (5.10), we get

$$P_S(\mathfrak{g}) = P_S(\mathfrak{p}) \oplus s(P_S(K)) \cong P_S(\mathfrak{p}) \oplus P_S(K)$$

From the discussion above we get

$$P_S(\mathfrak{g},\mathfrak{k}) \cong \mathfrak{T}(\mathfrak{g},K) \big/ S^+(\mathfrak{g})^{\mathfrak{g}}\cdot\mathfrak{T}(\mathfrak{g},K).$$

So the following diagram is a split exact sequence.

$$0 \longrightarrow P_S(\mathfrak{g},\mathfrak{k}) \hookrightarrow P_S(\mathfrak{g}) \underset{s}{\overset{\mathrm{pr}_{\mathfrak{k}}}{\twoheadrightarrow}} P_S(K) \longrightarrow 0 \tag{5.13}$$

Recall that the involution θ fixes $\mathfrak{k}$ and acts by -1 on $\mathfrak{p}$. It is easy to check that the transgression map $\mathsf{t}_{\wedge\mathfrak{g}^*}$ commutes with θ.

We claim that $P_S(\mathfrak{g})$ is stable under θ. Since we can choose a θ-stable triangular decomposition $\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$, the Harish–Chandra isomorphism can be assumed to be θ-equivariant. So it is enough to show that the space $P(W_{\mathfrak{g}})$ of Lemma 5.2 is θ-stable.

For $(\mathfrak{g},\mathfrak{k})=(\mathfrak{sl}(n),\mathfrak{so}(n))$ the action of θ on the space of diagonal matrices $\mathfrak{h}\subset\mathfrak{g}$ is given by minus the transpose with respect to the antidiagonal. (Here we are taking the realisation of $\mathfrak{k}=\mathfrak{o}(n)$ to be matrices that are skew-symmetric with respect to the antidiagonal.) Hence the even power sums are fixed by θ while the odd power sums are in the (-1)-eigenspace of θ. The claim follows.

For $(\mathfrak{g},\mathfrak{k})=(\mathfrak{sl}(2n),\mathfrak{sp}(2n))$, the involution θ is $X\mapsto -JX^tJ^{-1}=JX^tJ$, where J is the antidiagonal matrix having first n components equal to 1 and last n components equal to -1. As in the $(\mathfrak{sl}(n),\mathfrak{so}(n))$ case, the action of θ on the diagonal matrices comprising $\mathfrak{h}$ is given by minus the transpose with respect to the antidiagonal. Hence the same proof works also in this case.

Assume now that $(\mathfrak{g},\mathfrak{k})=(\mathfrak{e}(6),\mathfrak{f}(4))$. Let $\mu_1,\dots,\mu_l\in\mathfrak{h}^*$ be the weights of one of 27-dimensional irreducible $\mathfrak{e}(6)$-modules, then by [Oni, §11 Theorem 3] the polynomials

$$p_k=\sum_{j=1}^{l}\mu_j^{m_k+1},\qquad k=1,\dots,6,$$

where m_i are exponents of $\mathfrak{e}(6)$, form a basis of $P(W_{\mathfrak{g}})$. The involution θ which fixes $\mathfrak{f}(4)$ in $\mathfrak{e}(6)$ is induced by the outer automorphism. Therefore, $\theta(\mu_j)=-\mu_j$. Hence we get that

$$\theta(p_k)=\begin{cases}p_k & \text{if } m_k \text{ is odd},\\ -p_k & \text{if } m_k \text{ is even}.\end{cases}$$

Which proves that $P_S(\mathfrak{g})$ is θ-stable.

Set

$$P_\wedge(\mathfrak{g})^{\pm\theta}=\{p\in P_\wedge(\mathfrak{g})\mid\theta(p)=\pm p\}, \qquad P_S(\mathfrak{g})^{\pm\theta}=\{p\in P_S(\mathfrak{g})\mid\theta(p)=\pm p\},$$

We now consider the following diagram with the first row equal to (5.13).

$$\begin{array}{ccccc}
0 \longrightarrow P_S(\mathfrak{g},\mathfrak{k}) & \hookrightarrow & P_S(\mathfrak{g})=P_S(\mathfrak{g})^{\theta}\oplus P_S(\mathfrak{g})^{-\theta} & \xrightarrow{\mathrm{pr}_{\mathfrak{k}}} & P_S(K)\longrightarrow 0\\
\Big\downarrow \mathsf{t}_{(\wedge\mathfrak{p}^*)^K} & & \Big\downarrow \mathsf{t}_{\wedge\mathfrak{g}^*} & & \\
P_\wedge(\mathfrak{p}) & \xleftarrow[\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}]{} & P_\wedge(\mathfrak{g})=P_\wedge(\mathfrak{g})^{\theta}\oplus P_\wedge(\mathfrak{g})^{-\theta} & &
\end{array}$$

Clearly, θ acts by -1 on $\bigwedge^{\mathrm{odd}}\mathfrak{p}^*$. Since $P_\wedge(\mathfrak{g})$ is odd, $P_\wedge(\mathfrak{p})$ is odd too. Thus θ acts by -1 on $P_\wedge(\mathfrak{p})$, hence $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\,\mathsf{t}_{\wedge\mathfrak{g}^*}(P_S(\mathfrak{g})^{\theta})=0$ and $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\,\mathsf{t}_{\wedge\mathfrak{g}^*}(P_S(\mathfrak{g})^{-\theta})=P_\wedge(\mathfrak{p})$. Therefore, $\dim P_S(\mathfrak{g})^{-\theta}\geq\dim P_\wedge(\mathfrak{p})=\dim\mathfrak{a}$. Because $S(\mathfrak{k})$ is θ-invariant,

$P_S(\mathfrak{g})^{-\theta} \subset \ker \mathrm{pr}_{\mathfrak{k}} = P_S(\mathfrak{g},\mathfrak{k})$. Since we have seen that $\dim P_S(\mathfrak{g},\mathfrak{k}) = \dim \mathfrak{a}$, it follows that $P_S(\mathfrak{p}) = P_S(\mathfrak{g})^{-\theta}$. Therefore,

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \mathsf{t}_{\wedge \mathfrak{g}^*}(P_S(\mathfrak{g})) = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} \circ \mathsf{t}_{\wedge \mathfrak{g}^*}(P_S(\mathfrak{g})^{-\theta})$$

Since the left hand square commutes (up to multiplication by a non-zero scalar) by Lemma 4.14, we have $\mathsf{t}_{(\wedge \mathfrak{p}^*)^K}(P_S(\mathfrak{g},\mathfrak{k})) = P_\wedge(\mathfrak{p})$, hence (5.9) is an isomorphism. □

6. Harish–Chandra projections

In this section we construct relative analogues of the Harish–Chandra map $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{h})$. They are attached to elements of W^1 and denoted by hc_w. We show that for each $w \in W^1$, $\mathrm{hc}_w\colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$ is a surjective algebra homomorphim. We stress that in general it is not a filtered algebra homomorphism. However, in Section 7 we introduce a filtration on $\mathfrak{a}$ which makes hc_w compatible with filtrations.

Fix a triangular decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-.$$

By the Poincaré-Birkhoff-Witt Theorem, this decomposition of $\mathfrak{g}$ induces a decomposition of $\mathrm{Cl}(\mathfrak{g})$

$$\mathrm{Cl}(\mathfrak{g}) = \mathrm{Cl}(\mathfrak{h}) \oplus \left(\mathfrak{n}^- \,\mathrm{Cl}(\mathfrak{g}) + \mathrm{Cl}(\mathfrak{g})\mathfrak{n}^+\right),$$

which in turn defines a projection $\mathrm{hc}_{\mathfrak{g}}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$. By results of Kostant and Bazlov [Baz1, Theorem 4.1], see also [Mei, Theorem 11.8], the restriction of $\mathrm{hc}_{\mathfrak{g}}$ to $\mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}}$ is an algebra isomorphism

$$\mathrm{hc}_{\mathfrak{g}}\colon \mathrm{Cl}(\mathfrak{g})^{\mathfrak{g}} \to \mathrm{Cl}(\mathfrak{h}).$$

As it was shown in [AM4] the Harish-Chandra projection becomes a filtered algebra isomorphism when $\mathfrak{h}$ is given a special filtration, described in §8.

Assume now that $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ and fix a choice $\Delta^+(\mathfrak{k},\mathfrak{t})$ of positive $\mathfrak{t}$-roots in $\mathfrak{k}$ and a choice of $\Delta^+(\mathfrak{g},\mathfrak{t}) \supset \Delta^+(\mathfrak{k},\mathfrak{t})$. This determines the subset W^1 of the Weyl group $W_{\mathfrak{g}}$ (see the introduction).

Each $w \in W^1$ fixes a choice $\Delta^+_w(\mathfrak{g},\mathfrak{t})$ of positive $\mathfrak{t}$-roots in $\mathfrak{g}$ containing $\Delta^+(\mathfrak{k},\mathfrak{t})$. Consider the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+_w \oplus \mathfrak{h} \oplus \mathfrak{n}^-_w.$$

Set

$$\mathfrak{p}^+_w := \mathfrak{p} \cap \mathfrak{n}^+_w, \qquad \mathfrak{p}^-_w := \mathfrak{p} \cap \mathfrak{n}^-_w, \qquad \mathfrak{a} := \mathfrak{p} \cap \mathfrak{h},$$

so that

$$\mathfrak{p} = \mathfrak{p}^+_w \oplus \mathfrak{a} \oplus \mathfrak{p}^-_w.$$

Note that $\mathfrak{p}^+_w$ and $\mathfrak{p}^-_w$ are isotropic and in duality under B.

By the Poincaré-Birkhoff-Witt Theorem, there is a basis of $\mathrm{Cl}(\mathfrak{p})$ consisting of monomials

$$f_1^{i_1} \dots f_p^{i_p} h_1^{j_1} \dots h_a^{j_a} e_1^{k_1} \dots e_p^{k_p}, \tag{6.1}$$

with $f_1, \dots, f_p$ a basis of $\mathfrak{p}^-_w$, $h_1, \dots, h_a$ a basis of $\mathfrak{a}$, $e_1, \dots, e_p$ a basis of $\mathfrak{p}^+$, and the exponents i_r, j_s, k_t all equal to 0 or 1. This induces a decomposition

$$\mathrm{Cl}(\mathfrak{p}) = \mathrm{Cl}(\mathfrak{a}) \oplus \left(\mathfrak{p}^-_w \,\mathrm{Cl}(\mathfrak{p}) + \mathrm{Cl}(\mathfrak{p})\mathfrak{p}^+_w\right)$$

and consequently a Harish-Chandra projection $\mathrm{hc}_w\colon \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{a})$.

Lemma 6.2. *The restriction of the linear map* hc_w *to the* $\mathfrak{t}$*-invariants* $C(\mathfrak{p})^{\mathfrak{t}}$ *is an algebra homomorphism. Consequently,* $\mathrm{hc}_w\colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$ *is also an algebra homomorphism.*

Proof. If $x \in \mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}}$, then x can be written as sum of monomials (6.1) of $\mathfrak{t}$-weight 0. For the weight to be 0, each such monomial contains a factor in $\mathfrak{p}_w^+$ if and only if it contains a factor in $\mathfrak{p}_w^-$. It follows that

$$\mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}} \cap \left(\mathfrak{p}_w^- \,\mathrm{Cl}(\mathfrak{p}) + \mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+\right) = \mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}} \cap \left(\mathfrak{p}_w^- \,\mathrm{Cl}(\mathfrak{p})\right) = \mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}} \cap \left(\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+\right),$$

and this is clearly an ideal in $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}}$. Consequently, the projection $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p})^{\mathfrak{t}} \to \mathrm{Cl}(\mathfrak{a})$ is an algebra homomorphism. □

Lemma 6.3. *The Harish-Chandra projection* $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{a})$ *maps* $\alpha_{\mathfrak{p}}(U(\mathfrak{k}))$ *to* $\mathbb{C}$*.*

Proof. In writing the basis (6.1), we may assume that f_i and e_j are dual bases (with respect to B) of $\mathfrak{p}_w^-$ respectively $\mathfrak{p}_w^+$, such that each e_i is a root vector for the $(\mathfrak{g},\mathfrak{t})$-root β_i, while f_i is a root vector for the $(\mathfrak{g},\mathfrak{t})$-root $-\beta_i$. We can also assume that the h_i form an orthonormal basis for $\mathfrak{a}$.

Denote by $\rho_{\mathfrak{g}}$, respectively $\rho_{\mathfrak{k}}$, the half sum of roots in $\Delta^+(\mathfrak{g},\mathfrak{t})$, respectively $\Delta^+(\mathfrak{k},\mathfrak{t})$. The half sum of roots in $\Delta_w^+(\mathfrak{g},\mathfrak{t})$ is then $w\rho_{\mathfrak{g}}$. We claim that

$$\mathrm{hc}_w(\alpha(H)) = (w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}})(H), \qquad H \in \mathfrak{t}. \tag{6.4}$$

To see this, we compute

$$\begin{aligned}\alpha(H) &= \frac{1}{4}\Big(\sum_{i=1}^{p}[H,e_i]f_i + \sum_{i=1}^{a}[H,h_i]h_i + \sum_{i=1}^{p}[H,f_i]e_i\Big)\\ &\quad\text{(since } H\in\mathfrak{t} \text{ we have } [H,h_i]=0)\\ &= \frac{1}{4}\sum_{i=1}^{p}\beta_i(H)e_if_i - \frac{1}{4}\sum_{i=1}^{p}\beta_i(H)f_ie_i\\ &= \frac{1}{4}\sum_{i=1}^{p}\beta_i(H)(e_if_i - f_ie_i) = \frac{1}{4}\sum_{i=1}^{p}\beta_i(H)(2-2f_ie_i)\\ &= (w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}})(H) - \frac{1}{2}\sum_{i=1}^{p}\beta_i(H)f_ie_i.\end{aligned}$$

The last sum vanishes under the map hc_w by its definition and $(w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}})(H)$ stays unchanged by the same map because it is a constant. This proves (6.4).

Next, we claim that $\mathrm{hc}_w \circ\alpha$ annihilates elements of $\mathfrak{n}_w^+ \cap \mathfrak{k}$. By the definition of hc_w it is enough to show that for any root vector $x \in \mathfrak{n}_w^+ \cap \mathfrak{k}$, $\alpha(x) \in \mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+$. The summands in the expression for $\alpha(x)$ are of the form

$$[x,e_i]f_i \quad \text{or} \quad [x,h_i]h_i \quad \text{or} \quad [x,f_i]e_i.$$

Of these summands, $[x,f_i]e_i$ is clearly in $\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+$. Furthermore, $[x,h_i]h_i$ is in $\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+$, since $[x,h_i]$ is a positive root vector in $\mathfrak{p}$, hence in $\mathfrak{p}_w^+$, and it anticommutes with h_i. Finally, $[x,e_i]$ is a positive root vector in $\mathfrak{p}$, so it is in $\mathfrak{p}_w^+$, and it anticommutes with f_i, hence $[x,e_i]f_i$ is also in $\mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+$.

One shows similarly that if $y \in \mathfrak{n}_w^- \cap \mathfrak{k}$, then $\alpha(y) \in \mathfrak{p}_w^- \,\mathrm{Cl}(\mathfrak{p})$ (so $\mathrm{hc}_w(\alpha(y)) = 0$).

Let now M be a monomial in $U(\mathfrak{k})$ of the form

$$M = F_1 \cdots F_m H_1 \cdots H_k E_1 \cdots E_n,$$

where $F_1, \dots, F_m \in \mathfrak{n}_w^- \cap \mathfrak{k}$, $H_1, \dots, H_k \in \mathfrak{t}$, $E_1, \dots E_n \in \mathfrak{n}_w^+ \cap \mathfrak{k}$. By the Poincaré-Birkhoff-Witt Theorem, such monomials span $U(\mathfrak{k})$. We claim that $\mathrm{hc}_w(\alpha(M))$ is a constant; this will finish the proof.

If $n > 0$, then we can write $M = M'E_n$, so $\alpha(M) = \alpha(M')\alpha(E_n) \in \mathrm{Cl}(\mathfrak{p})\mathfrak{p}_w^+$, so $\mathrm{hc}_w(\alpha(M)) = 0$. Similarly, if $m > 0$, then $M = F_1M''$, so $\alpha(M) \in \mathfrak{p}_w^- \mathrm{Cl}(\mathfrak{p})$ and again $\mathrm{hc}_w(\alpha(M)) = 0$.

Finally, if $M = H_1 \cdots H_k$, then $\alpha(M) = \alpha(H_1)\dots\alpha(H_k)$ is $\mathfrak{t}$-invariant, so by Lemma 6.2 and by (6.4),

$$\mathrm{hc}_w(\alpha(M)) = \mathrm{hc}_w(\alpha(H_1))\dots\mathrm{hc}_w(\alpha(H_k)) = (w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}})(H_1)\dots(w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}})(H_k) \in \mathbb{C}.$$

□

Let e_i and f_i be dual bases for $\mathfrak{p}_w^+$, $\mathfrak{p}_w^-$ as before, let $p = \dim \mathfrak{p}_w^+$. Define

$$P_w = \frac{1}{2^p} e_1 \cdots e_p f_p \cdots f_1 \in \mathrm{Cl}(\mathfrak{p}).$$

It is easy to verify that P_w is a projection, i.e., $P_w^2 = P_w$; as we shall see, P_w is closely related to the Harish-Chandra map hc_w.

Lemma 6.5. *Let $\mathfrak{a}^+$ be a fixed maximal isotropic subspace of $\mathfrak{a}$ with respect to B. If $\mathfrak{p}$ is odd dimensional let $Z_1, Z_2 = \frac{1}{2}(1 \pm Z_{\mathrm{top}})$ be idempotents in the centre of $\mathrm{Cl}(\mathfrak{p})$ and $\underline{Z_1}, \underline{Z_2}$ be idempotents in the centre of $\mathrm{Cl}(\mathfrak{a})$. We construct $\mathfrak{p}^+$ to be $\mathfrak{p}_w^+ \oplus \mathfrak{a}^+$ and model S (resp. $S_1, S_2 = \bigwedge\mathfrak{p}^+ Z_1, \bigwedge\mathfrak{p}^+ Z_2$) by $\bigwedge(\mathfrak{p}^+)$. Then the projection P_w when considered in $\mathrm{End}(S)$ (resp $\mathrm{End}(S_1) \oplus \mathrm{End}(S_2)$) is the projection of S (resp. S_1, S_2) to the subspace*

$$e_{\mathrm{top}} \wedge \textstyle\bigwedge\mathfrak{a}^+ = e_1 \wedge \cdots \wedge e_p \wedge \bigwedge\mathfrak{a}^+ \quad (\textit{resp. } e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+ \underline{Z_1}, \quad e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+ \underline{Z_2})$$

along the span of monomials not containing $e_1 \wedge \cdots \wedge e_p$.

Proof. It is clear that P_w fixes every monomial in $e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+$, and kills every other monomial in $\bigwedge\mathfrak{p}^+$. The claim follows. □

Define the linear map $p_w : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{p})$ by $p_w(x) = P_w x P_w$. Since P_w commutes with $\mathrm{Cl}(\mathfrak{a})$, $p_w|_{\mathrm{Cl}(\mathfrak{a})}$ is a nonzero $\mathbb{Z}_2$-algebra morphism. The only $\mathbb{Z}_2$-ideals in $\mathrm{Cl}(\mathfrak{a})$ are $\mathrm{Cl}(\mathfrak{a})$ and $\{0\}$, thus $p_w|_{\mathrm{Cl}(\mathfrak{a})}$ is injective, and its image is clearly the subalgebra $P_w \mathrm{Cl}(\mathfrak{a}) P_w \subseteq \mathrm{Cl}(\mathfrak{p})$. Let $\phi_w : P_w \mathrm{Cl}(\mathfrak{a}) P_w \to \mathrm{Cl}(\mathfrak{a})$ be the inverse algebra isomorphism

$$\phi_w(P_w a P_w) = a, \quad a \in \mathrm{Cl}(\mathfrak{a}).$$

Lemma 6.6. *The Harish-Chandra projection hc_w is equal to $\phi_w \circ p_w$.*

Proof. Since $\mathfrak{p}_w^+ P_w = 0$ and $P_w \mathfrak{p}_w^- = 0$, the kernel of hc_w is contained in the kernel of p_w. The image of p_w is $P_w \mathrm{Cl}(\mathfrak{a}) P_w$, which has dimension $2^{\dim(\mathfrak{a})}$, just as $\mathrm{Cl}(\mathfrak{a}) = \mathrm{im}\,\mathrm{hc}_w$. It follows that $\ker p_w = \ker \mathrm{hc}_w$. On the other hand, $\phi_w \circ p_w$ and hc_w are both equal to the identity on $\mathrm{Cl}(\mathfrak{a})$, so they must be the same. □

Clearly, we can think of p_w as the projection of $\mathrm{End}(S)$ to $\mathrm{End}(e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+) \cong \mathrm{End}(\bigwedge\mathfrak{a}^+)$ (resp. $p_w : \mathrm{End}(S_1) \oplus \mathrm{End}(S_2) \to \mathrm{End}(e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+ \underline{Z_1}) \oplus \mathrm{End}(e_{\mathrm{top}} \wedge \bigwedge\mathfrak{a}^+ \underline{Z_2})$).

Proposition 6.7. *For each $w \in W^1$ the corresponding Harish–Chandra projection hc_w restricts to a surjective algebra homomorphism*

$$\mathrm{hc}_w \colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a}).$$

Proof. We already know (Lemma 6.2) that hc_w is an algebra homomorphism on $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$, so it remains to prove surjectivity. Since ϕ_w is an isomorphism, it is enough to show that p_w is surjective.

Assume first that $\mathrm{Cl}(\mathfrak{p}) = \mathrm{End}(S)$ (i.e., $\dim \mathfrak{p}$ is even). We want to show that any endomorphism of $V = e_{\mathrm{top}} \wedge \bigwedge \mathfrak{a}^+$ extends to a $\mathfrak{k}$-morphism of S, i.e., to an element of $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$. Let $v_1, \ldots, v_m$ be a basis of V, i.e., a basis of the space of highest weight vectors for the $\mathfrak{k}$-isotypic component of S corresponding to w. Let ψ_{ij} be the linear map on V sending v_i to v_j and v_k, $k \neq i$, to 0; it is enough to lift such maps since they span $\mathrm{End}(V)$. The highest weight vectors v_i and v_j generate isomorphic copies of the $\mathfrak{k}$-module $E_{w\rho_{\mathfrak{g}}-\rho_{\mathfrak{k}}}$ with highest weight $w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$, and there is a unique $\mathfrak{k}$-isomorphism of these two copies of $E_{w\rho_{\mathfrak{g}}-\rho_{\mathfrak{k}}}$ sending v_i to v_j. Now we extend this isomorphism by 0 to the copies of $E_{w\rho_{\mathfrak{g}}-\rho_{\mathfrak{k}}}$ generated by v_k, $k \neq i$, and to the other $\mathfrak{k}$-isotypic copies of S. In this way we get a $\mathfrak{k}$-endomorphism of S which is sent to ψ_{ij} by p_w.

The argument for $\dim \mathfrak{p}$ odd is similar. Using the same reasoning as above,

$$P_w \,\mathrm{Cl}(\mathfrak{a}) P_w = \mathrm{End}(e_{\mathrm{top}} \wedge \textstyle\bigwedge \mathfrak{a}^+ \underline{Z_1}) \oplus \mathrm{End}(e_{\mathrm{top}} \wedge \bigwedge \mathfrak{a}^+ \underline{Z_2}).$$

Now let $V_1 = e_{\mathrm{top}} \bigwedge \mathfrak{a}^+ \underline{Z_1}$ and $V_2 = e_{\mathrm{top}} \wedge \bigwedge \mathfrak{a}^+ \underline{Z_2}$. In an identical way to the even dimension case, given a linear map ψ_{ij} of either V_1 or V_2 we can lift this map to a $\mathfrak{k}$-module endomorphism of the Spin module $S_1 = \bigwedge \mathfrak{p}^+ Z_1$ and $S_2 = \bigwedge \mathfrak{p}^+ \wedge Z_2$. □

Remark 6.8. By Lemma 6.3, the Harish-Chandra homomorphism hc_w sends $\mathrm{Pr}(S) = \alpha(Z(\mathfrak{k}))$, see Definition 2.11, to constants. It is easy to see that in terms of projections, hc_w sends pr_w to 1, and other pr_σ to 0.

Indeed, since all elements of $e_{\mathrm{top}} \wedge \bigwedge \mathfrak{a}^+$ are of weight $w\rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$, this space is contained in the $\mathfrak{k}$-isotypic component of S corresponding to w. Thus pr_w is the identity on this space, so $p_w(\mathrm{pr}_w) = P_w$ and therefore $\mathrm{hc}_w(\mathrm{pr}_w) = \phi_w(P_w) = 1$. On the other hand, for $\sigma \neq w$, pr_σ kills the space $e_{\mathrm{top}} \wedge \bigwedge \mathfrak{a}^+$, so $p_w(\mathrm{pr}_\sigma) = 0$ and thus also $\mathrm{hc}_w(\mathrm{pr}_\sigma) = 0$.

7. Clifford algebra analogue of Cartan's theorem for primary and almost primary cases

In this section we prove an analogue of Cartan's theorem for $\mathrm{Cl}(\mathfrak{p})^K$ for primary and almost primary cases. Let $(\mathfrak{g}, \mathfrak{k}, K)$ be a triple from the list (5.1).

Lemma 7.1. *Assume that* $(\mathfrak{g}, \mathfrak{k}, K) = (\mathfrak{sl}(2n), \mathfrak{o}(2n), O(2n))$. *In this case* $|W^1| = 2$. *Let* hc_1 *and* hc_2 *be the corresponding Harish–Chandra homomorphisms and* pr_1, pr_2 *be the corresponding projections. Then the restrictions of* hc_1 *and* hc_2 *to* $\mathrm{Cl}(\mathfrak{p})^K$ *coincide and define an isomorphism* $\mathrm{hc}\colon \mathrm{Cl}(\mathfrak{p})^K \to \mathrm{Cl}(\mathfrak{a})$.

Proof. Let $k = \mathrm{diag}(1, \ldots, 1, -1)$, so k is a representative of the disconnected component of K. Let $\tilde{k}$ be a lift of k in the pin double cover $\tilde{K}$ of K. The two $\mathfrak{k}$-types in S differ only by the sign of the last coordinate of the highest weight. The action of $\tilde{k}$ switches these two highest weights because the action of k on $\mathfrak{h}$ changes the sign of the last coordinate. This implies that $\mathrm{Ad}_k \mathrm{pr}_1 = \mathrm{pr}_2$ and therefore $\mathrm{hc}_1 = \mathrm{hc}_2 \circ \mathrm{Ad}_k$. Thus for $\omega \in \mathrm{Cl}(\mathfrak{p})^K$, $\mathrm{hc}_1(\omega) = \mathrm{hc}_2(\omega)$ since $\mathrm{Ad}_k \omega = \omega$.

Suppose that $\mathrm{hc}(\omega) = 0$. Note that $\omega = \omega \,\mathrm{pr}_1 + \omega \,\mathrm{pr}_2$, $\mathrm{hc}_1(\omega \,\mathrm{pr}_2) = 0$ and $\mathrm{hc}_2(\omega \,\mathrm{pr}_1) = 0$, so it follows that $\mathrm{hc}_i(\omega \,\mathrm{pr}_i) = 0$. Since hc_i is injective on $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \,\mathrm{pr}_i$, it follows that $\omega \,\mathrm{pr}_i = 0$. So $\omega = \omega \,\mathrm{pr}_1 + \omega \,\mathrm{pr}_2 = 0$. This proves injectivity of $\mathrm{hc}_i\colon \mathrm{Cl}(\mathfrak{p})^K \to \mathrm{Cl}(\mathfrak{a})$ and surjectivity follows by dimension counting. □

Proposition 7.2. *For any $w \in W^1$ the restriction of hc_w to $\mathrm{Cl}(\mathfrak{p})^K$ is an algebra isomorphism. Moreover, the restrictions of all hc_w to $\mathrm{Cl}(\mathfrak{p})^K$ coincide.*

Proof. For the primary cases we have that $|W^1| = 1$ and $\dim \mathrm{Cl}(\mathfrak{a}) = \dim \mathrm{Cl}(\mathfrak{p})^K = 2^{\dim \mathfrak{a}}$, so the claim follows from Proposition 6.7 which states hc_w is surjective. The claim for the remaining almost primary cases $(\mathfrak{sl}(2n), \mathfrak{o}(2n), O(2n))$ follows from the Lemma above. □

Definition 7.3. The space of primitive invariants $P_{\mathrm{Cl}}(\mathfrak{p})$ in $\mathrm{Cl}(\mathfrak{p})^K$ is defined as $q(P_\wedge(\mathfrak{p}))$, where $P_\wedge(\mathfrak{p})$ is as in Definition 5.5.

It follows from the relative transgression Theorem 5.8 and Lemma 4.18 that $P_{\mathrm{Cl}}(\mathfrak{p}) = \mathrm{im}\, \underline{\mathfrak{t}}_{\mathrm{Cl}(\mathfrak{p})^K}$.

Theorem 7.4. *For each $w \in W^1$ the Harish Chandra projection $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p}) \to \mathrm{Cl}(\mathfrak{a})$, takes primitives $\phi \in P_{\mathrm{Cl}}(\mathfrak{p})$ to linear elements of $\mathrm{Cl}(\mathfrak{a})$, i.e., to $q(\bigwedge^1 \mathfrak{a})$. Moreover, $\mathrm{hc}_w(P_{\mathrm{Cl}}(\mathfrak{p})) = \mathfrak{a}$.*

Proof. Lemma 6.3 shows that $\mathrm{hc}_w : \mathrm{im}\, \alpha_\mathfrak{p} \to \mathbb{C}$. Theorem 4.20 states that we have $\iota_x \phi \in \mathrm{im}\, \alpha_\mathfrak{p}$ for all $x \in \mathfrak{p}$, hence $\mathrm{hc}_w\, \iota_x \phi \in \mathbb{C}$. The linear map hc_w commutes with contractions by elements from $\mathfrak{a}$ hence

$$\iota_x\, \mathrm{hc}_w(\phi) = \mathrm{hc}_w(\iota_x \phi) \in \mathbb{C} \quad \forall x \in \mathfrak{a}.$$

Since ϕ is odd, this implies that $\mathrm{hc}_w(\phi)$ is linear.

The fact that $\mathrm{hc}_w(P_{\mathrm{Cl}}(\mathfrak{p})) = \mathfrak{a}$ follows from two facts:
1) $\dim P_{\mathrm{Cl}}(\mathfrak{p}) = \dim \mathfrak{a}$ by (5.6),
2) $\mathrm{hc}_w \colon \mathrm{Cl}(\mathfrak{p})^K \to \mathrm{Cl}(\mathfrak{a})$ is an algebra isomorphism by Proposition 7.2. □

Corollary 7.5. *For any two elements $\phi, \psi \in P_{\mathrm{Cl}}(\mathfrak{p})$, the commutator $[\phi, \psi]$ is in $\mathbb{C}$.*

Proof. Theorem 7.4 shows that for every $w \in W^1$ the Harish Chandra map hc_w takes elements of $P_{\mathrm{Cl}}(\mathfrak{p})$ to linear terms. For any $a_1, a_2 \in \mathfrak{a} \subset \mathrm{Cl}(\mathfrak{a})$, $[a_1, a_2] \in \mathbb{C}$, hence the primitives $P_\wedge(\mathfrak{p})$ have the same property in $\mathrm{Cl}(\mathfrak{p})^K$, since the Harish Chandra map is an isomorphism of algebras $\mathrm{hc}_w : \mathrm{Cl}(\mathfrak{p})^K \to \mathrm{Cl}(\mathfrak{a})$. □

7.1. The form on primitives in $\mathrm{Cl}(\mathfrak{p})^K$.

For $X \in \mathrm{Cl}(\mathfrak{p})$, let X^L be the operator of left-multiplication by X, and X^R the operator of ($\mathbb{Z}_2$-graded) right-multiplication:

$$X^L(Y) = XY, \qquad X^R(Y) = (-1)^{p(X)p(Y)} YX.$$

for parity homogeneous elements $X, Y \in \mathrm{Cl}(\mathfrak{p})$, and $X^L - X^R = [X, -]_{\mathrm{Cl}}$. If $x \in \mathfrak{p}$, we have

$$x^L = q \circ (\varepsilon_x + \iota_x) \circ q^{-1}, \qquad x^R = q \circ (\varepsilon_x - \iota_x) \circ q^{-1},$$

where ε_x is the operator of left-multiplication by x in $\bigwedge \mathfrak{p}$.

Lemma 7.6. *Let $X \in \bigwedge^m \mathfrak{p}$, $Y \in \bigwedge^n \mathfrak{p}$, then*

$$q^{-1}([q(X), q(Y)]) \in \bigoplus_{k=|m-n|}^{m+n} \bigwedge^k \mathfrak{p}.$$

Moreover, if $m \geq n$ then $q^{-1}([q(X), q(Y)])_{[m-n]} = (1 - (-1)^m)\iota_X Y$.

Proof. Recall that $[q(X), q(Y)] = (q(X)^L - q(X)^R)q(Y)$.

Assume that $X = x_1 \wedge \ldots \wedge x_m$, where $x_1, \ldots, x_m \in \mathfrak{p}$. Then

$$q(X)^L(q(Y)) = q \circ (\varepsilon_{x_1} + \iota_{x_1}) \circ \ldots \circ (\varepsilon_{x_m} + \iota_{x_m})(Y)$$

When expanding out the brackets $(\epsilon_{x_i} + \iota_{x_i})$ we find that the resulting operator $q(X)^L$ is a summand of monomials with k multiplication operators and $m - k$ contraction operators, for $k \in \{0, \cdots, m\}$. Since $\deg \varepsilon_{x_i} = 1$ and $\deg \iota_{x_i} = -1$, then $q(X)^L(q(Y)) \in \bigoplus_{k=n-m}^{m+n} \bigwedge^k \mathfrak{p}$, a similar argument shows that $q(X)^R(q(Y)) \in \bigoplus_{k=n-m}^{m+n} \bigwedge^k \mathfrak{p}$, thus

$$q^{-1}([q(X), q(Y)]) \in \bigoplus_{k=n-m}^{m+n} \bigwedge^k(\mathfrak{p}).$$

On the other hand, we have that

$$[q(X), q(Y)] = (-1)^{mn}[q(Y), q(X)] = (-1)^{mn}(q(Y)^L - q(Y)^R)q(X).$$

Thus by an identical argument considering operators $q(X)^L$ and $q(X)^R$ we find

$$q^{-1}([q(X), q(Y)]) \in \bigoplus_{k=m-n}^{m+n} \bigwedge^k(\mathfrak{p}).$$

Thus $q^{-1}([q(X), q(Y)])$ is concentrated between degrees $|n-m|$ and $n+m$. When $m \geq n$, the $m - n$th degree term of $[q(X), q(Y)]$ is equal to

$$q(X)^L(q(Y))_{[n-m]} - q(X)^R(q(Y))_{[n-m]} = \iota_X(Y) - (-1)^m \iota_X(Y).$$

Thus $[q(X), q(Y)]_{[n-m]} = (1 - (-1)^m)q(\iota_X(Y))$. □

Lemma 7.7. *Suppose two odd homogeneous elements $X, Y \in \bigwedge \mathfrak{p}$ are such that $[q(X), q(Y)] \in \mathbb{C}$. Then $[q(X), q(Y)] = 2(\iota_X Y)_{[0]}$.*

Proof. Suppose that $X \in \bigwedge^m \mathfrak{p}$, $Y \in \bigwedge^n \mathfrak{p}$ and $n \neq m$. Then $(\iota_X Y)_{[0]} = 0$. On the other hand, by Lemma 7.6,

$$q^{-1}([q(X), q(Y)]) \in \bigoplus_{k=|m-n|}^{m+n} \bigwedge^k \mathfrak{p}.$$

In particular, since $|m - n| > 0$ we have that $q^{-1}([q(X), q(Y)])_{[0]} = 0$. By assumption we get that $[q(X), q(Y)] = q^{-1}([q(X), q(Y)])_{[0]} = 0 = 2(\iota_X Y)_{[0]}$.

Now suppose that $X, Y \in \bigwedge^m \mathfrak{p}$ have the same degree, then

$$q^{-1}(q(X)^L q(Y))_{[0]} = \iota_X Y, \qquad q^{-1}(q(X)^R q(Y))_{[0]} = (-1)^m \iota_X Y.$$

Hence we get that

$$\begin{aligned} q^{-1}([q(X), q(Y)])_{[0]} &= q^{-1}(q(X)^L q(Y))_{[0]} - q^{-1}(q(X)^R q(Y))_{[0]} \\ &= \iota_X Y - (-1)^m \iota_X Y = (1 - (-1)^m)\iota_X Y = (1 - (-1)^m)(\iota_X Y)_{[0]}. \end{aligned}$$

Since m was assumed to be odd, we conclude that $[q(X), q(Y)] = 2(\iota_X Y)_{[0]}$. □

Recall that $\tilde{B} : \mathrm{Cl}(\mathfrak{p}) \times \mathrm{Cl}(\mathfrak{p}) \to \mathbb{C}$ is the form defined on elements $a, b \in \mathrm{Cl}(\mathfrak{p})$ by

$$\tilde{B}(a, b) = \iota_{q^{-1}(a)}(b)_{[0]}.$$

By abuse of notation, we denote the restriction of the form $\tilde{B}$ on $P_{\mathrm{Cl}}(\mathfrak{p})$ again by $\tilde{B}$. Applying Lemma 7.7 and Corollary 7.5 to the primitives $P_{\mathrm{Cl}}(\mathfrak{p})$ we find:

Corollary 7.8. *Let* $\phi, \psi \in P_\wedge(\mathfrak{p})$ *be homogeneous elements. Then*

$$[q(\phi), q(\psi)] = 2(\iota_\phi \psi)_{[0]} = 2\tilde{B}(\phi, \psi).$$

7.2. **Proof of Theorem 1.1 for primary and almost primary symmetric pairs.** To prove statement (a) of Theorem 1.1, note first that the elements of $P_{\mathrm{Cl}}(\mathfrak{p}) \subset \mathrm{Cl}(\mathfrak{p})^K$ satisfy the relations from Corollary 7.5. Thus it follows from the universal property of $\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B})$ that the inclusion $P_\wedge(\mathfrak{p}) \hookrightarrow \mathrm{Cl}(\mathfrak{p})^K$ extends to an algebra homomorphism

$$f\colon \mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B}) \to \mathrm{Cl}(\mathfrak{p})^K.$$

Furthermore, Theorem 7.4(a) shows that for every $w \in W^1$

$$\mathrm{hc}_w(P_{\mathrm{Cl}}(\mathfrak{p})) \subseteq \mathfrak{a} \subset \mathrm{Cl}(\mathfrak{a}).$$

By Proposition 7.2 we have that $\mathrm{hc}_w|_{\mathrm{Cl}(\mathfrak{p})^K}$ is an isomorphism and $\mathrm{hc}_w(P_\wedge(\mathfrak{p})) = \mathfrak{a} \subset \mathrm{Cl}(\mathfrak{a})$. We conclude $\mathrm{hc}_w(P_{\mathrm{Cl}}(\mathfrak{p}))$ generates $\mathrm{Cl}(\mathfrak{a})$ and $P_{\mathrm{Cl}}(\mathfrak{p})$ generates $\mathrm{Cl}(\mathfrak{p})^K$. In particular,

$$\mathrm{hc}_w \circ f\colon \mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B}) \to \mathrm{Cl}(\mathfrak{a})$$

is an algebra homomorphism and $\mathrm{hc}_w(f(P_{\mathrm{Cl}}(\mathfrak{p}))) = \mathfrak{a}$. The restriction of the form $\tilde{B}$ on $P_{\mathrm{Cl}}(\mathfrak{p})$ must therefore be non-degenerate. Indeed, suppose that a nonzero $\phi \in P_{\mathrm{Cl}}(\mathfrak{p})$ is orthogonal to $P_{\mathrm{Cl}}(\mathfrak{p})$, then ϕ anticommutes in $\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}), \tilde{B})$ with all of $P_{\mathrm{Cl}}(\mathfrak{p})$ (including itself), so the nonzero element $\mathrm{hc}_w(f(\phi)) \in \mathfrak{a}$ anticommutes with $\mathfrak{a}$ (including itself). But there is no such element in $\mathfrak{a}$.

(b) When $(\mathfrak{g}, \mathfrak{k})$ is primary, $\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p})) \cong \mathrm{Cl}(\mathfrak{p})^K$ and $\mathrm{im}\,\alpha_Z = \mathrm{Pr}(S)$ is $\mathbb{C}$ (see Lemma 2.12). So there is nothing left to prove. In the almost primary cases, $\mathrm{im}\,\alpha_Z = \mathrm{Pr}(S)$ is two dimensional spanned by pr_1 and pr_2. It is clear that $\mathrm{im}\,\alpha_Z$ is in the centre of $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$ thus there is a filtered algebra homomorphism from $\mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p})) \otimes \mathrm{Pr}(S)$ to $\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}}$. We are left to show that this map is injective.

Let $a_1, a_2 \in \mathrm{Cl}(P_{\mathrm{Cl}}(\mathfrak{p}))$ be such that $a_1 \otimes \mathrm{pr}_1 + a_2 \otimes \mathrm{pr}_2$ is in the kernel, then taking the Harish Chandra projection hc_i shows that $a_i = 0$, thus this algebra homormorphism is an injection. Dimension counting shows it is a filtered algebra isomorphism.

(c) This is proved in [CNP] and [CGKP] for any symmetric pair.

Statement (d) follows from the fact that $\mathrm{gr}\,\mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \cong (\bigwedge\mathfrak{p})^{\mathfrak{k}}$, $\mathrm{gr}(\mathrm{Cl}(P_\wedge(\mathfrak{p})) \cong \bigwedge P_\wedge(\mathfrak{p})$, $\mathrm{gr}\,\mathbb{C}[\mathfrak{t}^*]^{W_\mathfrak{k}}/I_\rho \cong \mathbb{C}[\mathfrak{t}^*]^{W_\mathfrak{k}}/I_+$ and $\mathrm{gr}(A \otimes B) \cong \mathrm{gr}\,A \otimes \mathrm{gr}\,B$. ☐

Remark 7.9. In a similar manner, starting from Theorem 3.57 and the Harish–Chandra projection $\mathrm{hc}_\mathfrak{g}\colon \mathrm{Cl}(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{h})$, one can prove Kostant's result that $\mathrm{Cl}(\mathfrak{g})^\mathfrak{g}$ equals $\mathrm{Cl}(q(P_\wedge(\mathfrak{g})), \tilde{B})$ without using the Hodge theory for $\bigwedge\mathfrak{g}$.

7.3. **Contractions by primitive elements.** Let $x, a, b \in \mathrm{Cl}(\mathfrak{p})$ then

$$\begin{aligned}[x, ab] &= xab - (-1)^{|x||a|+|x||b|}ab \\ &= xab - (-1)^{|x||a|}axb + (-1)^{|x||a|}axb - (-1)^{|x||a|+|x||b|}ab \\ &= [x,a]b + (-1)^{|x||a|}a[x,b].\end{aligned}$$

That is, $[x, -]$ is a derivation of $\mathrm{Cl}(\mathfrak{p})$.

Let $(\mathfrak{g}, \mathfrak{k}, K)$ be a triple from the list (5.1).

Corollary 7.10. *Let* $\phi \in P_\wedge(\mathfrak{p})$*, then the restriction of* ι_ϕ *to* $\mathrm{Cl}(\mathfrak{p})^K$ *is equal to* $\frac{1}{2}[q(\phi), -]$ *and hence is a derivation of* $\mathrm{Cl}(\mathfrak{p})^K$*.*

Proof. Let ϕ_i be a basis of $P_\wedge(\mathfrak{p})$, orthonormal with respect to $\tilde{B}$, and such that $\phi_i \in (\bigwedge^{m_i}\mathfrak{p})^K$. For $1 \leq i_1 < i_2 < \ldots < i_k \leq \dim P_\wedge(\mathfrak{p})$, set $\psi = \phi_{i_1} \wedge \ldots \wedge \phi_{i_k} \in (\bigwedge^M \mathfrak{p})^K$, where $M = m_{i_1} + \ldots + m_{i_k}$. Since ϕ_i are orthonormal, Theorem 1.1(a) implies that $q(\psi) = q(\phi_{i_1}) \ldots q(\phi_{i_k})$.

Using the fact that $[q(\phi_j), -]$ is a derivation of $\mathrm{Cl}(\mathfrak{p})$, we have that

$$[q(\phi_j), q(\psi)] = \sum_a (-1)^{a-1} q(\phi_{i_1}) \ldots [q(\phi_j), q(\phi_{i_a})] \ldots q(\phi_{i_k})$$

(Since $[q(\phi_a), q(\phi_b)] = 2\delta_{a,b}$ and all i_a are distinct.)

$$= \begin{cases} 2(-1)^{a-1} q(\phi_{i_1}) \ldots \widehat{q(\phi_{i_a})} \ldots q(\phi_{i_k}) & \text{if there exists } a \text{ such that } i_a = j, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\widehat{q(\phi_{i_a})}$ denotes the omission of $q(\phi_{i_a})$. We have that $q^{-1}([q(\phi_j), q(\psi)]) \in \left(\bigwedge^{M-m_j}\mathfrak{p}\right)^K$. By Lemma 7.6 we get that

$$q^{-1}([q(\phi_j), q(\psi)]) = q^{-1}([q(\phi_j), q(\psi)])_{[M-m_i]} = 2\iota_{\phi_j}\psi.$$

Thus $\frac{1}{2}[q(\phi_j), -]$ and ι_{ϕ_j} agree on every monomial of the form $q(\phi_{i_1}) \ldots q(\phi_{i_k})$ which form a basis in $\mathrm{Cl}(\mathfrak{p})^K$, since they are both linear these operators agree on $\mathrm{Cl}(\mathfrak{p})^K$. □

Corollary 7.11. *For $\phi \in P_\wedge(\mathfrak{p})$ the restriction of ι_ϕ to $(\bigwedge\mathfrak{p})^K$ is a derivation of $(\bigwedge\mathfrak{p})^K$.*

Proof. By Corollary 7.10, for $\psi \in P_\wedge(\mathfrak{p})$ the operator ι_ϕ is a derivation of $\mathrm{Cl}(\mathfrak{p})^K$. The associated graded map of $\iota_\phi \in \mathrm{End}(\mathrm{Cl}(\mathfrak{p}))$ is $\iota_\phi \in \mathrm{End}(\bigwedge\mathfrak{p})$ when we pass from $\mathrm{Cl}(\mathfrak{p})$ to the associated graded $\bigwedge\mathfrak{p}$. Since ι_ϕ is a derivation of $\mathrm{Cl}(\mathfrak{p})^K$ it preserves $\mathrm{Cl}(\mathfrak{p})^K$, hence its graded version ι_ϕ preserves $(\bigwedge\mathfrak{p})^K$. Moreover, the associated graded algebra to $\mathrm{Cl}(\mathfrak{p})^K$ is $(\bigwedge\mathfrak{p})^K$. The associated graded map of a derivation is a derivation of the associated graded algebra, thus we get that ι_ϕ is a derivation of $(\bigwedge\mathfrak{p})^K$. □

8. An analogue of Kostant's Clifford algebra conjecture.

In this section we define a filtration on $\mathfrak{a}$ which makes the Harish–Chandra projections $\mathrm{hc}_w \colon \mathrm{Cl}(\mathfrak{p})^{\mathfrak{k}} \to \mathrm{Cl}(\mathfrak{a})$ filtered algebra morphisms. This generalises Kostant's Clifford algebra conjecture to the relative case.

We first review some background material about principal $\mathfrak{sl}_2$-triples, for example, see [Dyn, Kos1, Vog]. Let $\mathfrak{g}$ be a semisimple Lie algebra. Up to conjugacy, there is a finite number of Lie algebra embeddings $i \colon \mathfrak{sl}_2 \to \mathfrak{g}$. An $\mathfrak{sl}_2$-embedding is called *principal* if $\mathfrak{g}$ splits into the smallest number of $i(\mathfrak{sl}_2)$-submodules under the adjoint action. Recall that by the Hopf–Koszul–Samelson Theorem $P_\wedge(\mathfrak{g})$ is a graded vector space with generators in degrees $2m_i + 1$ defined by the exponents $m_1, \ldots, m_r$ of $\mathfrak{g}$

Let $\check{\mathfrak{g}}$ be the Lie algebra defined by the dual root system and $\check{\mathfrak{h}} \subset \check{\mathfrak{g}}$ be the corresponding Cartan subalgebra. Kostant observed that $\rho_{\mathfrak{g}} \in \mathfrak{h}^*$ viewed as an element of $\check{\mathfrak{h}}$ coincides with the regular element $\check{h}$ of the principal $\mathfrak{sl}_2$-embedding to $\check{\mathfrak{g}}$ given by $(\check{e}, \check{h}, \check{f}) \subset \check{\mathfrak{g}}$. Since $\check{\mathfrak{h}}^*$ and $\mathfrak{h}$ are canonically isomorphic, the coadjoint action of $\check{e}$ induces a filtration on $\mathfrak{h}$:

$$\mathcal{F}^{(m)}\mathfrak{h} = \left\{ x \in \mathfrak{h} \mid (\mathrm{ad}^*_{\check{e}})^{m+1} x = 0 \right\}.$$

The dimension of the vector space $\mathcal{F}^{(m)}\mathfrak{h}$ jumps at values $m = m_1, \ldots, m_r$ which follows from results of [Kos1].

Since $\mathrm{hc}_{\mathfrak{g}}(q(P_\wedge(\mathfrak{g}))) = \mathfrak{h}$, we can define a different filtration on $\mathfrak{h}$:

$$F^{(m)}\mathfrak{h} = \mathrm{hc}_{\mathfrak{g}}(q(P_\wedge^{(2m+1)}(\mathfrak{g}))),$$

where $P_\wedge^{(k)}(\mathfrak{g})$ denotes the filtration on $P_\wedge(\mathfrak{g})$ induced by the $\mathbb{Z}$-grading. Kostant suggested the following conjecture which was proved for type A in [Baz2] and for any reductive Lie algebra in [Jos, AM4].

Theorem 8.1 (Kostant's Clifford algebra conjecture)**.** *Two filtrations on* $\mathfrak{h}$ *coincide, namely, for any* $m \in \mathbb{N}$*, we have* $\mathcal{F}^{(m)}\mathfrak{h} = F^{(m)}\mathfrak{h}$*.*

Similarly, for any $w \in W^1$ define two filtrations on $\mathfrak{a}$ by

$$\mathcal{F}^{(m)}\mathfrak{a} = \left\{ x \in \mathfrak{a} \mid (\mathrm{ad}^*_{\check{e}})^{m+1} x = 0 \right\}, \quad F^{(m)}\mathfrak{a} = \mathrm{hc}_w(P_\wedge^{(2m+1)}(\mathfrak{p})),$$

where $P_\wedge^{(k)}(\mathfrak{p})$ denotes the filtration on $P_\wedge(\mathfrak{p})$ induced by the $\mathbb{Z}$-grading. Note that the filtration $F^{(m)}\mathfrak{a}$ does not depend on w since by Proposition 7.2 we have that the image of $P_\wedge(\mathfrak{p})$ under hc_w does not depend on w.

Theorem 8.2 (Relative Kostant's Clifford algebra conjecture)**.** *Let* $(\mathfrak{g}, \mathfrak{k})$ *be a symmetric pair of primary or almost primary type, then the two filtrations on* $\mathfrak{a}$ *coincide, namely, for any* $m \in \mathbb{N}$*, we have* $\mathcal{F}^{(m)}\mathfrak{a} = F^{(m)}\mathfrak{a}$*.*

Proof. Given that $P_\wedge(\mathfrak{p}) = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}(P_\wedge(\mathfrak{g}))$ then the filtration $F^{(m)}\mathfrak{a}$ is just the restriction of $F^{(m)}\mathfrak{h}$ to $\mathfrak{a}$. Furthermore, since $\mathfrak{a} \subset \mathfrak{h}$, $\mathcal{F}^{(m)}\mathfrak{a}$ is the restriction of $\mathcal{F}^{(m)}\mathfrak{h}$ to $\mathfrak{a}$. Thus the theorem follows from restriction to $\mathfrak{a}$ of the results from Theorem 8.1; the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}^{(m)}\mathfrak{h} & \xrightarrow{\cong} & F^{(m)}\mathfrak{h} \\ \Big\downarrow {\scriptstyle \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}} & & \Big\downarrow {\scriptstyle \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}} \\ \mathcal{F}^{(m)}\mathfrak{a} & \xrightarrow{\cong} & F^{(m)}\mathfrak{a}. \end{array}$$

Which concludes the proof. □

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(K. Calvert) Department of Mathematics and Statistics, Lancaster University, Bailrigg, Lancaster, UK
Email address: kieran.calvert@lancaster.ac.uk

(K. Grizelj) Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
Email address: karmen.grizelj@math.hr

(A. Krutov) Mathematical Institute of Charles University, Sokolovská 83, 186 75 Prague, Czech Republic
Email address: andrey.krutov@matfyz.cuni.cz

(P. Pandžić) Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
Email address: pandzic@math.hr