

A Lagrangian Klein bottle you can't squeeze

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Dedicated to Claude Viterbo, on his fifty-tenth birthday.

Abstract

Suppose you have a nonorientable Lagrangian surface L in a symplectic 4-manifold. How far can you deform the symplectic form before the smooth isotopy class of L contains no Lagrangians? I solve this question for a particular Lagrangian Klein bottle. I also discuss some related conjectures.

1 Introduction

Here are two overlapping questions:

Question 1.1 (Minimal nonorientable genus). Given a symplectic 4-manifold (X, ω) and a $\mathbb{Z}/2$ -homology class $\beta \in H_2(X; \mathbb{Z}/2)$, what is the minimal nonorientable genus of a nonorientable Lagrangian surface $L \subset X$ with $[L] = \beta$?

Question 1.2 (Nonsqueezing). Given a symplectic 4-manifold (X, ω) and a nonorientable Lagrangian surface $L \subset X$, how far can you deform ω in cohomology before there is no Lagrangian smoothly isotopic to L ?

If L is orientable then these questions are less interesting: the genus is determined by $[L]^2 = -\chi(L)$ and, in Question 1.2, it is necessary to deform ω subject to the cohomological condition $\int_L [\omega] = 0$. By contrast, if L is nonorientable, we have $H^2(L; \mathbb{R}) = 0$, which means that it is possible to deform ω , keeping L Lagrangian, in such a way that $[\omega]$ ranges over an open set in $H^2(X; \mathbb{R})$.

I will give some general discussion of these questions in turn, then give a concrete example of a Lagrangian Klein bottle for which Question 1.2 can be answered completely (Theorem 3.1).

One running theme throughout the discussion is the use of *visible* and *tropical* Lagrangians in almost toric 4-manifolds: these provide a rich source of Lagrangian submanifolds coming respectively from straight lines and tropical curves in integral affine surfaces. I have found them useful for thinking about some of the phenomena under discussion, and for formulating conjectures. Visible Lagrangians were introduced in Symington's work [16]; tropical Lagrangians were introduced independently by Mikhalkin [9] and Matessi [8].

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2 The minimal genus question

2.1 Review

Definition 2.1. Define the *nonorientable genus* of the nonorientable surface $\#_k \mathbf{RP}^2$ to be k . Proposition 1.1 of [4] shows that any $\mathbb{Z}/2$ -homology class in a symplectic 4-manifold can be represented by some embedded nonorientable Lagrangian, so Question 1.1 has a well-defined answer, which I will denote¹ by $\eta(X, \omega, \beta)$.

Remark 2.2. Audin [1] showed that

$$P_2(\beta) = \chi(L) = 2 - k \pmod{4},$$

where P_2 denotes the Pontryagin square operation and χ is the Euler characteristic. If you find a Lagrangian with nonorientable genus k then you can perform a Hamiltonian finger move locally to introduce pairs of intersections with index difference 1 and then perform Polterovich surgery [13] on these self-intersections to get an embedded Lagrangian with nonorientable genus $k + 4$. This means that the set of genera which can be realised is $\{\eta(X, \omega, \beta), \eta(X, \omega, \beta) + 4, \dots\}$.

Remark 2.3. The quantity $\eta(X, \omega, \beta)$ is known in a small range of cases, the lower bound being the principal difficulty.

1. When X satisfies $[\omega] \cdot c_1(X) > 0$, we know that $\eta(X, \omega, 0) = 6$. This follows from Givental's construction [5] of a Lagrangian $\#_6 \mathbf{RP}^2$ in the 4-ball and from the fact, proved by Shevchishin [14] that X contains no nullhomologous Lagrangian Klein bottles (see also the beautiful papers by Nemirovski [12, 11]).
2. Let $X_{a,b,c}$ be the blow-up of the 4-ball in three subballs so that the symplectic areas of the exceptional spheres E_1, E_2, E_3 are a, b, c . Shevchishin and Smirnov [15] show that $E_1 + E_2 + E_3$ contains a Lagrangian \mathbf{RP}^2 if and only if the following inequalities all hold

$$a < b + c, \quad b < c + a, \quad c < a + b.$$

They call these the *symplectic triangle inequalities*. This gives the lower bound $\eta(X_{a,b,c}, \omega, E_1 + E_2 + E_3) \geq 5$ when a, b, c violate the triangle inequalities.

¹ η is the International Phonetic Alphabet symbol for the “ng” sound.

Remark 2.4. After the fact, we see that there is a *tropical* or *almost toric* motivation for the Shevchishin-Smirnov triangle inequalities. The almost toric base diagram in Figure 1 depicts the blow-up $X_{a,b,c}$; the affine lengths a, b, c indicated correspond to the sizes of the exceptional spheres E_1, E_2, E_3 . In red you can see a tropical curve; using the ideas of Mikhalkin [9] and Matessi [8], we can construct a Lagrangian submanifold L living over a (small thickening of a) tropical curve. This tropical Lagrangian is diffeomorphic to \mathbf{RP}^2 if and only if the inequalities all hold: the preimage of the point marked with cross-hairs is a circle in L whose neighbourhood is a Möbius strip.

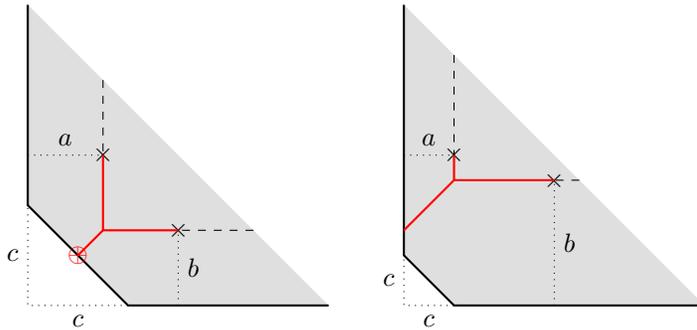


Figure 1: Almost toric base diagrams for $X_{a,b,c}$ with a tropical curve in red. Left: The symplectic triangle inequalities and the associated tropical Lagrangian is diffeomorphic to \mathbf{RP}^2 (with the core circle of a cross-cap living over the point marked by the cross-hair symbol). Right: The symplectic triangle inequalities are violated and the associated tropical Lagrangian is diffeomorphic to a disc.

2.2 $S^2 \times S^2$

Let $X = S^2 \times S^2$. Modulo an overall scale factor, any symplectic form on X is diffeomorphic to one from the family $\lambda p_1^* \sigma + p_2^* \sigma$, where $p_1, p_2: X \rightarrow S^2$ are the two projections and σ is an area form on S^2 . We know that $\eta(X, \omega, 0) = 6$, which leaves two interesting $\mathbb{Z}/2$ -homology classes up to diffeomorphism: $\beta = [\star \times S^2]$ and the class Δ of the diagonal. The Pontryagin squares are $P_2(\beta) = 0$ and $P_2(\Delta) = 2$, so there is a chance to represent β by Lagrangian Klein bottles.

Lemma 2.5. *If $\lambda < 2$ then β is represented by a Lagrangian Klein bottle.*

Proof. The rectangle in Figure 2 is the toric moment polygon for the standard Hamiltonian torus action on $S^2 \times S^2$ with symplectic form ω_λ . There is a Lagrangian Klein bottle living over the line ℓ (slope 1/2) in the diagram. To see this, consider the two S^2 factors sitting inside \mathbb{R}^3 and let (p_j, θ_j) be cylindrical coordinates on the j th factor ($j = 1, 2$). These are action-angle coordinates, so $\omega = \sum dp_j \wedge d\theta_j$. The line ℓ is given by $2p_2 = p_1$ and the Lagrangian Klein

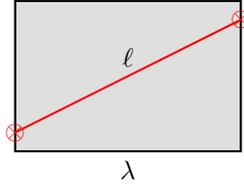


Figure 2: A visible Lagrangian Klein bottle in $(S^2 \times S^2, \omega_\lambda)$ for $\lambda < 2$. The cores of two cross-caps are indicated with cross-hairs.

bottle is cut out by this equation together with $\theta_2 = -2\theta_1$. This is certainly Lagrangian for this symplectic form. To see that L is a Klein bottle, notice that the regular level sets of p_1 restricted to L are circles $\theta_2 = -2\theta_1$ in the (θ_1, θ_2) -torus, which collapse 2-to-1 onto the circles of maxima and minima at $p_1 = \pm\lambda$ (as the torus collapses to the circle with coordinate θ_2). The projections of these circles are denoted with cross-hairs in Figure 2. \square

Remark 2.6. This L is a *visible Lagrangian* in the sense of Symington [16] as well as being a tropical Lagrangian in the sense of Matessi [8] and Mikhalkin [9]. This Klein bottle is well-known: it appears in [3] as a Hamiltonian minimal Lagrangian, in [6] as a Hamiltonian suspension, and in [4] as a fibre connect-sum of \mathbf{RP}^2 s. It has minimal Maslov number 1 and has a monotone representative in its Lagrangian isotopy class if $\lambda = 1$.

If $\lambda \geq 2$ then the line ℓ does not fit into the rectangle. The following conjecture seems natural; while I cannot prove it, it inspired Theorem 3.1 below.

Conjecture 2.7. *There is no Lagrangian Klein bottle in the class β if $\lambda \geq 2$.*

It is interesting to consider what happens for large λ . We have essentially no tools to prove lower bounds when the Lagrangians are of high genus and may be Floer-theoretically obstructed. The most pessimistic conjecture is that Lagrangians with high genus become flexible enough that:

Conjecture 2.8. $\lim_{\lambda \rightarrow \infty} \eta(X, \omega_\lambda, \beta) < \infty$.

The following lemma gives an upper bound on $\eta(X, \omega_\lambda, \beta)$, but it goes to infinity with λ .

Lemma 2.9. *We have $\eta(X, \omega_\lambda, \beta) \leq 20\ell + 2$ when $\lambda < 10\ell + 2$.*

Proof. If $\lambda < 10\ell + 1$ then there is a tropical Lagrangian in the class β with nonorientable genus $20\ell + 2$. We show the tropical curve for $\ell = 2$ in Figure 3 below; for general ℓ we simply repeat the pattern between the vertical blue bars as often as required to get from the left-hand side to the right-hand side of the rectangle.

The edges of this tropical curve are:

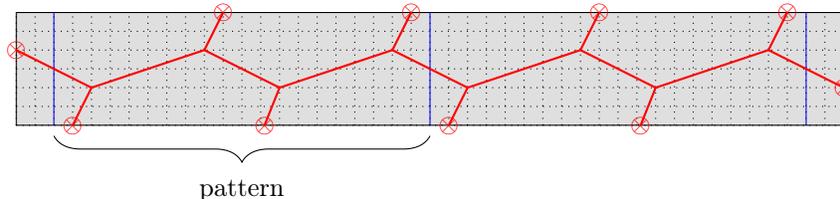


Figure 3: A tropical curve giving a Lagrangian of genus $20\ell + 2$ in the case $\ell = 2$.

- internal edges parallel to either $(3, 1)$ or $(2, -1)$,
- external edges parallel to $(2, -1)$ or $(1, 2)$.

The corresponding tropical Lagrangian intersects the horizontal spheres with even multiplicity and the vertical spheres with odd multiplicity, so it inhabits the class β . The vertices of the tropical curve are not smooth²: each has self-intersection equal to 2. By [9, Theorem 3.2], this tropical curve therefore yields an immersed Lagrangian with 8ℓ double points and $2 + 4\ell$ cross-caps where it hits the boundary (marked with cross-hairs in Figure 3). When we perform Polterovich surgery at the double points, we obtain a Lagrangian which is topologically a surface of genus 8ℓ with $4\ell + 2$ cross-caps. This has Euler characteristic $2 - 16\ell - 4\ell - 2 = -20\ell$, so the nonorientable genus is $2 + 20\ell$. \square

Remark 2.10. It seems harder to make the genus significantly smaller using tropical Lagrangians, but there is no reason to believe that tropical Lagrangians should give a sharp upper bound for η .

3 Nonsqueezing

3.1 Statement

For each connected open interval $I \subset \mathbb{R}$ (length $|I|$), let C_I denote the cylinder $I \times (\mathbb{R}/2\pi\mathbb{Z})$ with coordinates (p, θ) , equipped with the symplectic form $\frac{1}{2\pi} dp \wedge d\theta$; this has total area $|I|$. Let S^2 denote the 2-sphere equipped with its area form σ satisfying $\int_{S^2} \sigma = 2$.

Let $U_I = S^2 \times C_I$. Note that U_I is obtained from $(S^2 \times S^2, \omega_{|I|})$ by excising the spheres $S^2 \times \{n, s\}$, where n, s denote the poles of the second factor. Arguing as in Lemma 2.5, we see that if $|I| > 1$, the only nontrivial class $\beta \in H_2(U_I; \mathbb{Z}/2)$ is represented by a Lagrangian Klein bottle (see Figure 4).

²At each vertex of a tropical curve, the outgoing edges v_1, v_2, v_3 must sum to zero; if we write m for the determinant $|v_1 \wedge v_2| = |v_2 \wedge v_3| = |v_3 \wedge v_1|$ then the self-intersection of this vertex is defined to be $\frac{m-1}{2}$. Smoothness means all vertices have self-intersection zero.

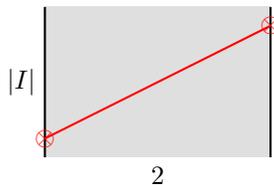


Figure 4: The visible Lagrangian Klein bottle in U_I when $|I| > 1$.

Theorem 3.1. *Suppose that $|I| \leq 1$. If $\iota: K \rightarrow U_I$ is a Lagrangian embedding of the Klein bottle in the class β then $\iota_*: \mathbb{Q} = H_1(K; \mathbb{Q}) \rightarrow H_1(U_I; \mathbb{Q}) = \mathbb{Q}$ is the zero map.*

Remark 3.2. The proof of Theorem 3.1 will occupy the rest of the paper. It uses SFT and neck-stretching.

Remark 3.3. Note that if $|I| > 1$ then $H_1(L; \mathbb{Q}) \rightarrow H_1(U_I; \mathbb{Q})$ is an isomorphism for the Lagrangian Klein bottle L coming from Lemma 2.5. To see this, take either one of the circles living over the points marked with cross-hairs in Figure 4; this is a generator for both $H_1(L; \mathbb{Q})$ and $H_1(U_I; \mathbb{Q})$. We deduce:

Corollary 3.4. *The Lagrangian Klein bottle in $U_{(0,1+\epsilon)}$ from Lemma 2.5 cannot be squeezed into $U_{(0,1)}$.*

Remark 3.5. To reduce Conjecture 2.7 to this result, you would need to produce a pair of symplectic spheres in the class $[S^2 \times \star]$ which “link” your Lagrangian Klein bottle in an appropriate way. Since this class has non-minimal symplectic area, it is difficult to control the SFT limit of such spheres.

We now proceed to the proof of Theorem 3.1.

3.2 Mohnke’s almost complex structure

Pick a flat metric g on the Klein bottle. There is a contact form (the canonical 1-form) on the unit cotangent bundle $M \subset T^*K$ whose closed Reeb orbits correspond to closed geodesics on K . We will not distinguish notationally between geodesics and the corresponding Reeb orbits and we will write $-\gamma$ for the geodesic obtained by reversing γ . There are two isolated simple geodesics γ_0, γ_1 which are the core circles for two disjoint embedded Möbius strips in K . Any isolated geodesic is a multiple cover of one of these and all other geodesics occur in one-parameter families. We call the isolated geodesics *odd* and the other geodesics *even*.

Theorem 3.6 (Mohnke [10, Section 2.1]). *There exists an almost complex structure J^- on the cotangent bundle T^*K with the following properties:*

1. J^- is cylindrical at infinity and suitable for neck-stretching.

2. For any geodesic γ there is a finite-energy J^- -holomorphic cylinder f_γ in T^*K asymptotic to γ and $-\gamma$.
3. [10, Lemma 7(2)] Any J^- -holomorphic cylinder in T^*K which intersects the zero-section is one of these f_γ for some closed geodesic γ .

Remark 3.7. If we let $W := \overline{T^*K}$ denote the compactification of the cotangent bundle obtained by gluing on its ideal contact boundary M then there is a well-defined intersection pairing $H_2(W, M; \mathbb{Z}/2) \otimes H_2(W; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2$. The cylinders f_γ define elements of $H_2(W, M; \mathbb{Z}/2)$ and we have [10, Lemma 7(3)]

$$f_\gamma \cdot K = \begin{cases} 1 & \text{if } \gamma \text{ is odd} \\ 0 & \text{if } \gamma \text{ is even.} \end{cases}$$

Remark 3.8 ([10, Lemma 7(1)]). Note that there are also no finite energy planes in T^*K , nor in the symplectisation $\mathbb{R} \times M$, for *any* cylindrical almost complex structure adapted to our chosen contact form. This is because there are no contractible Reeb orbits, and a finite energy plane would provide a nullhomotopy of its asymptote.

3.3 Neck-stretching

Let $I = (0, 1)$ and $\bar{I} = [0, 1]$. Suppose there is a Lagrangian Klein bottle $K \subset U_I$ such that $\mathbb{Q} = H_1(K; \mathbb{Q}) \rightarrow H_1(U_I; \mathbb{Q}) = \mathbb{Q}$ is nonzero (in particular, it is injective). Think of K sitting inside $U_{\bar{I}}$ and make symplectic cuts to $U_{\bar{I}}$ at $p = 0, 1$ to obtain a Lagrangian Klein bottle K living in the manifold $X = S^2 \times S^2$ equipped with the product symplectic form giving the factors areas 2 and 1 respectively. Crucially, the symplectic cut introduces symplectic spheres S_0 and S_1 (at the $p = 0, 1$ cuts respectively) which are disjoint from K .

Pick a sequence of almost complex structures J_t , $t \in \mathbb{R}$, on X with the following properties:

- on a Weinstein neighbourhood of K , J_t coincides with Mohnke's almost complex structure J^- ;
- on a neck-stretching region $(a_t, b_t) \times M$ around K , J_t is a neck-stretching sequence;
- the spheres S_0, S_1 are J_t -holomorphic for all $t \in \mathbb{R}$.

Pick a point k on K which does not lie on any of the cylinders f_γ for an odd geodesic γ . Let $u_t: S^2 \rightarrow X$ be a J_t -holomorphic curve representing the class $\alpha = [\star \times S^2]$ and such that $u_t(0) = k$; there is a unique such u_t up to reparametrisation by a theorem of Gromov [7, 2.4.C], since α is a minimal area sphere class in X .

By the SFT compactness theorem [2], there is a sequence t_i such that u_{t_i} converges (after reparametrisations) to a holomorphic building with components

in T^*K (the completion of the Weinstein neighbourhood of K), components in $\mathbb{R} \times M$ (the completion of the neck) and components in $X \setminus K$ (the completion of the complement of the Weinstein neighbourhood).

3.4 SFT limit analysis

The components v_1, \dots, v_n of the SFT limit building living in $X \setminus K$ can be compactified, yielding topological surfaces in X with boundary on K ; we will still denote these by v_1, \dots, v_n . The sum of the ω -areas of the v_i (weighted by multiplicities if the SFT limit involves a branched cover) equals the ω -area of α , which is 1.

Lemma 3.9. *There must be at least two planar components amongst the v_i , possibly geometrically indistinct (i.e. having the same image).*

Proof. First note that the limit building intersects K because $u_t(0) = k \in K$ for all t . It also necessarily has at least one component in $X \setminus K$ because T^*K is exact and so contains no closed holomorphic curves. A genus zero holomorphic building with at least two levels must have two planar components (just for topological reasons) though these could be geometrically indistinct. Any planar components live in $X \setminus K$. \square

Lemma 3.10. *There are two components v_0, v_1 of the limit building such that $v_i \cdot S_j = \delta_{ij}$. These components are planar and there are no further components of the limit building in $X \setminus K$.*

Proof. Since α intersects S_0 and S_1 there must be components of the limit building which intersect S_0 and S_1 . By positivity of intersections, either:

- (A) there is one component v_1 which hits both S_0 and S_1 once transversely and all other components are disjoint from S_0, S_1 .
- (B) there are two components v_0, v_1 such that v_0 intersects S_0 once transversely and is disjoint from S_1 and vice versa for v_1 .

Moreover, each of these components occurs with multiplicity one in the SFT limit in order to get the correct intersection numbers $\alpha \cdot S_0, \alpha \cdot S_1$.

If v_2 is a component which does not intersect S_0 or S_1 then it defines a class in $H_2(U_I, K; \mathbb{Z})$. By assumption, the kernel of the map $\mathbb{Z} \oplus \mathbb{Z}/2 = H_1(K; \mathbb{Z}) \rightarrow H_1(U_I; \mathbb{Z}) = \mathbb{Z}$ is precisely the torsion part. Therefore the long exact sequence

$$\cdots \rightarrow H_2(U_I; \mathbb{Z}) \rightarrow H_2(U_I, K; \mathbb{Z}) \rightarrow H_1(K; \mathbb{Z}) \rightarrow H_1(U_I; \mathbb{Z}) \rightarrow \cdots$$

splits off a sequence

$$\cdots \rightarrow \mathbb{Z} \rightarrow H_2(U_I, K; \mathbb{Z}) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

This implies that the areas of classes in $H_2(U_I, K; \mathbb{Z})$ are half-integer multiples of the area of the generator $\beta \in H_2(U_I; \mathbb{Z})$, which is 2. Therefore v_2 has integer area.

The area of α is 1, so the (weighted) sum of the ω areas of the v_i equals 1. Since v_1 has positive area, v_2 must have positive area strictly less than 1, but this is not possible if v_2 has integer area. Therefore there cannot be any component v_2 disjoint from S_0 and S_1 .

By Lemma 3.9, there are at least two planar components (or one planar component with multiplicity two) in the limit building. This is not compatible with Case (A), so we must be in Case (B) and v_0, v_1 must additionally be planes. \square

Lemma 3.11. *1. All the remaining parts of the limit building are cylinders.*

- 2. At least one of these cylinders lives in T^*K and has the form f_γ for an odd geodesic γ .*
- 3. There are no other cylindrical components of the SFT limit building in T^*K .*

Proof. 1. If a component has three or more punctures then the limit building must contain at least three planar components (counted with multiplicity) but we have seen that all the planar components must live in $X \setminus K$ (Remark 3.8) and that there are precisely two such components (Lemma 3.10).

2. Since $u_t(0) = k$ for all t , the limit building contains a component in T^*K , which must be a cylinder of the form f_γ by Theorem 3.6(3). At least one of these cylindrical components must correspond to an odd geodesic because α has odd intersection with K in $H_2(X; \mathbb{Z}/2)$ and the intersection number picks up contributions from each component of the building inside T^*K , which are nontrivial if and only if γ is odd (Remark 3.7).

3. If there are two or more cylindrical components in T^*K then there must be a further cylindrical component in $\mathbb{R} \times M$ which connects the asymptotes of two of these cylinders. Since this cylinder has no positive asymptote, this cannot exist by the maximum principle. \square

Proof of Theorem 3.1. We chose $k \in K$ not to lie on any of the cylinders f_γ for γ an odd geodesic, but we have showed that these are the only cylinders which can arise as components of the SFT limit building. Since the SFT limit building must pass through k , we get a contradiction. \square

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