Newtonian Potential and Geodesic Completeness in Infinite Derivative Gravity

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Recent study has shown that a non-singular oscillating potential – a feature of Infinite Derivative Gravity (IDG) theories – matches current experimental data better than the standard GR potential. In this work we show that this non-singular oscillating potential can be given by a wider class of theories which allows the defocusing of null rays, and therefore geodesic completeness. We consolidate the conditions whereby null geodesic congruences may be made past-complete, via the Raychaudhuri Equation, with the requirement of a non-singular Newtonian potential in an IDG theory. In doing so, we examine a class of Newtonian potentials characterised by an additional degree of freedom in the scalar propagator, which returns the familiar potential of General Relativity at large distances.

I.   INTRODUCTION

In the century since Albert Einstein ushered in a new paradigm for modern physics by formulating a gravitational theory that is described by the curvature of spacetime, General Relativity (GR) has withstood numerous experimental tests [1]. formulating a gravitational theory In the following century, GR It accurately describes our universe all the way down to short distances (5.6×10^{-5}m from a source [2]) [1] and has been found to be in agreement with experimental tests on gravitational redshift and the equivalence principle [3]. More recently, the detection of gravitational waves lends more weight to this already colossal theory [4]. However, that is not to say that GR does not have any shortcomings. At the classical level, the theory breaks down at short distances in its description of black holes, for instance. GR also cannot be made geodesically past-complete, when an appropriate energy condition is met, indicating the presence of an initial singularity in the theory [5–9].

In a geodesically-incomplete spacetime, causal geodesic congruences converge to a point in a finite 'time' (affine parameter). In such a scenario a freely falling particle or photon will simply cease to exist in a finite 'time', which suggests a serious physical malady in the theory [10]. These shortcomings allow us to consider GR to be a first approximation to a broader theory.

This notion of *extending* GR, via additional curvature terms in the gravitational action, forms the basis for many modified theories of gravity.

Significant examples include local theories such as $f(R)$-gravity, which replaces the curvature scalar in the Einstein-Hilbert action with an arbitrary function $f(R)$, or Stelle's 4th derivative gravity, which can be seen as a generalisation of the Gauss-Bonnet term in that it includes tensorial as well as scalar modifications to the Einstein-Hilbert action. However, these finite higher-derivative models still come hand-in-hand with a number of weaknesses. Stelle's theory, while perturbatively renormalizable [11], suffers from the introduction of *ghosts* - physical excitations, characterised by negative kinetic energy [12]. $f(R)$-gravity, in comparison, may avoid the introduction of ghosts [3] but breaks down at short distances, meaning that the theory can not be said to be UV-complete [13]. In contrast, Infinite Derivative Gravity (IDG) offers a means of 'completing' GR in the UV regime (short distances). IDG is characterised by an action containing an infinite series of d'Alembertian operators ($\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$) acting on the curvature along with the mass scale of the theory, M.

IDG is inspired by similar infinite derivative actions used in string theory [14]. The action was first applied to gravity in [15], where the form of the modified propagator was also calculated around a Minkowski background, see also [16, 17]. The full non-linear equations of motion were computed in [18] and the boundary terms found in [19]. The gravitational entropy for the IDG action was investigated around an (A)dS metric in [20], while the form of the (A)dS propagator was given in [21]. Constraints were put on the mass scale M of IDG, either by looking at the deflection of light around the Sun [22], or by using data on the tensor-scalar ratio and spectral tilt of the Cosmic Microwave Background [23]. It has been shown that in an inflationary framework, IDG can be thought of as an extension to Starobinsky inflation [24–27].

In [28, 29], it was shown using a toy model of IDG that it is possible to curtail the divergences of 1-loop diagrams and show that 2-loop diagrams are finite, while in [30] the UV finiteness of IDG theories were investigated. Further work has focused on the Newtonian potential around a flat background [15, 31, 32], formulating the Hamiltonian of the IDG action [33] and avoiding cosmological singularities both through using the ansatz $\Box R = c_1 R + c_2$ [25, 34] and via the Raychaudhuri equation [35–37].

In this paper we investigate the Newtonian potential $\Phi(r)$, describing the gravitational field of a small static spherically symmetric test mass in a flat space background. In General Relativity, this diverges according to $\Phi(r) \sim -1/r$, becoming singular at the origin. IDG offers a means of resolving this divergence.

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[1] Dark energy notwithstanding

It was shown in [18] that the equations of motion for IDG can be formulated in terms of two arbitrary functions of the d'Alembertian operator $a(\Box)$ and $c(\Box)$, which also characterise the modification to the gravitational propagator. In the case where these functions are equal, i.e. $a(\Box) = c(\Box)$, no additional degrees of freedom other than the massless graviton enter the system. Previous work [14, 15, 31] has shown that, in this case, a non-singular Newtonian potential can be derived, where the potential takes the form of an error function $\Phi \sim -\mathrm{Erf}(Mr/2)/r$, for the simplest choice of $a(\Box) = e^{-\Box/M^2}$.

If we generalise this further by taking $a(\Box)$ to be an exponential of a higher order polynomial, the potential is modified by an oscillating function at these higher orders. Analysis by Perivolaropoulos [2, 38] has shown that this oscillating function provides a better fit to experimental data on the force of gravity at small distances than the standard GR theory. Although further analysis is needed, there are tantalising hints that modified gravity could provide the correct description of the strength of gravity at small distances.

However, work on the avoidance of singularities in IDG theories [35–37, 39] has shown that we require a departure from the simple choice of $a(\Box) = c(\Box)$ in order for causal geodesic congruences to be made past-complete. without requiring exotic matter. The aim of the present work is to consolidate the requirements of a non-singular theory of gravity, known as the *defocusing* conditions, with the aforementioned converging Newtonian potential. To this end, we examine a wider class of Newtonian potentials, characterised by the condition $a(\Box) \neq c(\Box)$, in tandem with the defocusing conditions derived in [36] around a Minkowski background. This would allow us to avoid singularities by permitting the defocusing of null rays in a theory with a well-defined Newtonian potential at short distances.

In Section II, we give an overview of Infinite Derivative Gravity (IDG), and show how the theory offers a means of rendering null rays geodesically-complete. In Section III, we derive the Newtonian potential for IDG, and specifically look at the case which allows defocusing. Finally in Section IV we plot the Newtonian potential and interpret the results.

II. INFINITE DERIVATIVE GRAVITY

As mentioned in the introduction, ghosts are physical excitations bearing negative kinetic energy. These excitations are represented by a negative residue in the gravitational propagator. When interactions in such a system take place, the vacuum decays into both positive and negative energy states. This is known as the Ostrogradsky instability [41].

Previous attempts to resolve singularities by modifying gravity, such as Stelle's 4th derivative gravity [11], resulted in the introduction of ghosts, where the Hamiltonian of the theory was unbounded from below due to the Ostrogradsky instability [41]. By adding an infinite number of derivatives to the theory this instability may be avoided through an appropriate choice of the functions $a(\Box)$ and $c(\Box)$. In [33] it was shown that the Hamiltonian of IDG is indeed bounded from below.

The IDG action, which is the most general, torsion-free and parity invariant action of gravity, that is quadratic in curvature was first derived in [15, 39]

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\Big(M_p^2 R + R F_1(\Box) R$$
$$+ R^{\mu\nu} F_2(\Box) R_{\mu\nu} + C^{\mu\nu\lambda\sigma} F_3(\Box) C_{\mu\nu\lambda\sigma} \Big), \quad (1)$$

where R is the Ricci curvature scalar, $R_{\mu\nu}$ is the Ricci tensor, $C_{\mu\nu\lambda\sigma}$ is the Weyl tensor and M_p is the Planck mass. Each $F_i(\Box) = \sum_{n=0}^{\infty} f_{i_n} \Box^n / M^{2n}$ is a function of the d'Alembertian operator $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. M is the scale of modification of our theory and the f_{i_n} are the dimensionless coefficients of the series.

The equations of motion for IDG around a Minkowski background are given by [36]

$$\kappa T_{\mu\nu} = a(\Box) R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} c(\Box) R - \frac{1}{2} f(\Box)\nabla_\mu\nabla_\nu R, \ (2)$$

where we have defined[2]

$$a(\Box) = 1 + M_P^{-2}\Big(F_2(\Box) + 2F_3(\Box) \Big)\Box,$$
$$c(\Box) = 1 + M_P^{-2}\left(-4F_1(\Box) - F_2(\Box) + \frac{2}{3}F_3(\Box) \right)\Box,$$
$$f(\Box) = M_P^{-2}\left(4F_1(\Box) + 2F_2(\Box) + \frac{4}{3}F_3(\Box) \right), \quad (3)$$

which abide by the constraint $a(\Box) - c(\Box) = f(\Box)\Box$. From (2) we can derive the propagator around a flat background [15, 16]

$$\Pi(p^2) = \frac{\mathcal{P}^{(2)}}{p^2\, a(-p^2)} - \frac{\mathcal{P}_s^{(0)}}{p^2(a(-p^2) - 3c(-p^2))} \quad (4)$$

where $\mathcal{P}^{(2)}$ is the spin-2 projection operator, $\mathcal{P}_s^{(0)}$ is the scalar spin-0 projection operator and $p_\mu = i\partial_\mu$ is the momentum, so that in flat space $\Box = -g^{\mu\nu}p_\mu p_\nu = -p^2$. The simplest choice is to set $a(\Box) = c(\Box)$ equal to the exponential of an entire function, which by definition does not have any roots, and therefore the propagator will not receive any additional degrees of freedom other than the massless graviton. However, it was shown in [39], see also [17, 37], that the scalar sector of the propagator can have at most one additional pole without ghost-like degrees of

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[2] We denote the linearised curvatures around a Minkowski background as R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$.

freedom being introduced. This allows us to write the ghost-free condition [3]

$$(a(\Box) - 3c(\Box))R = 2\left(\Box/m^2 - 1\right)\bar{a}(\Box)R, \qquad (5)$$

where $\bar{a}(\Box)$ is the exponential of an entire function and so $a(-p^2) - 3c(-p^2)$ has a single pole in the scalar sector at $\Box \to -p^2 = m^2$, which produces an extra spin-0 particle of mass m. By expanding to first order in $\Box$, we find that $m^2 \equiv M_P^2/(6f_{1_0} - f_{2_0} - M_p^2/M^2)$ [36]. We know that $m^2 > 0$ so that the particle has real mass and therefore no tachyons are introduced.

If either $a(\Box) = c(\Box)$ or (5) holds, the propagator contains an exponential term in the denominator, which suppresses the propagator at high momentum. This exponential suppression results in an exponential enhancement of the vertex factors of the relevant Feynman diagrams [17, 28, 29, 39] meaning that interaction does not take place at this point, as in a local theory. In this sense, the action can be considered to describe an effective theory.

A. Defocusing conditions

The Raychaudhuri equation is a model-independent identity which relates the geometry of spacetime to the contribution of gravity via the curvature. It says that for a null tangent vector k_μ, satisfying $k^\mu k_\mu = 0$, the expansion parameter $\theta = \nabla_\mu k^\mu$ is described by[4]

$$\frac{d\theta}{d\lambda} + \frac{1}{2}\theta^2 \leq -R_{\mu\nu}k^\mu k^\nu, \qquad (6)$$

where λ is the affine parameter [10]. In order to have an expansion parameter which is positive and increasing, and therefore allow the defocusing of the null rays, we require $R_{\mu\nu}k^\mu k^\nu < 0$. In GR, using the Einstein equation, $G_{\mu\nu} = \kappa T_{\mu\nu}$, we cannot fulfil this condition because the Null Energy Condition (NEC) requires that $T_{\mu\nu}k^\mu k^\nu \geq 0$. By the Hawking-Penrose Singularity Theorem [40], this inability to defocus will always lead to a singularity. To be precise, a spacetime cannot be null geodesically complete in the past direction if $R_{\mu\nu}k^\mu k^\nu > 0$ [9].

However, in IDG this is not the case and we will see that it is possible to have defocusing. By contracting the equations of motion (2) with $k^\mu k_\nu$, we can see that the contribution of gravity to the Raychaudhuri equation is

$$R_{\mu\nu}k^\mu k^\nu = \frac{1}{a(\Box)}\left[\kappa T_{\mu\nu}k^\mu k^\nu + \frac{k^\mu k^\nu}{2}f(\Box)\nabla_\mu\nabla_\nu R\right], (7)$$

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[3] $(a(\Box) - 3c(\Box))R$ corresponds to the trace equation, found by contracting (2) with $g^{\mu\nu}$.

[4] We have taken the simplest case of the Raychaudhuri equation here by making two simplifications. Firstly, we take the congruence of null rays to be orthogonal to the hypersurface, so that the twist tensors vanish. Secondly, the shear tensor gives a positive contribution to the right-hand side and so we can neglect it for our purposes.

which was studied in a cosmological setting [36]. When considering a static spherically symmetric perturbation around a flat background, then the defocusing condition is

$$R_{\mu\nu}k^\mu k^\nu = \frac{1}{a(\Box)}\left[\kappa T_{\mu\nu}k^\mu k^\nu + \frac{(k^r)^2}{2}f(\Box)\Box R(r)\right] < 0, \ (8)$$

where $\Box R(r) = \frac{1}{r^2}\partial_r(r^2\partial_r)R(r)$. Note that the function $a(\Box)$ acting on the curvature cannot be negative, as this would lead to the Weyl ghost. This is because $a(-p^2)$ is the modification to the spin-2 part of the propagator, so it must be positive to avoid negative residues [16]. Therefore if the NEC holds true, we arrive at the minimum defocusing condition

$$\frac{a(\Box) - c(\Box)}{a(\Box)}R(r) < 0. \qquad (9)$$

The first observation to make is that defocusing cannot occur in the case of $a(\Box) = c(\Box)$, which corresponds to requiring that there are no extra poles in the scalar propagator and therefore no extra scalar propagating modes. As above, we can allow a single extra scalar propagating mode without introducing ghosts into the system [17, 37, 39]. This corresponds to adding an extra root to $a(\Box) - 3c(\Box)$ which from (5) means that the relationship between $a(\Box)$ and $c(\Box)$ is given by [36]

$$c(\bar{\Box}) = \frac{a(\bar{\Box})}{3}\left[1 + 2\left(1 - \Box/m^2\right)\tilde{a}(\bar{\Box})\right], \qquad (10)$$

where $\tilde{a} \equiv \frac{\bar{a}(\Box)}{a(\Box)}$ is an exponent of an entire function. Hence the propagator is given by [36]

$$\Pi(p^2) = \frac{1}{a(-p^2)}\left[\frac{\mathcal{P}^{(2)}}{p^2} + \frac{\mathcal{P}_S^{(0)}}{2p^2(1 + p^2/m^2)\tilde{a}(-p^2)}\right], \ (11)$$

while the minimum condition for null rays to defocus (9) becomes

$$(1 - \Box/m^2)\tilde{a}(\Box)R(r) > R(r). \qquad (12)$$

III. NEWTONIAN POTENTIAL

In this section, we describe the general method for finding the Newtonian Potential, which was previously studied for the more specific case when $a(\Box) = c(\Box)$ [15, 31, 42], and we will generalise that to the case when $a(\Box) \neq c(\Box)$. When we take the metric generated by a small static spherically symmetric test mass added to a flat space background, following the method of [15, 31, 42],

$$ds^2 = -\left(1 + 2\Phi(r)\right)dt^2 + \left(1 - 2\Psi(r)\right)\eta_{ij}dx^i dx^j, (13)$$

this is akin to perturbing the flat space metric $\eta_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where

$$h_{00} = h^{00} = -2\Phi \quad \text{and} \quad h_{ij} = h^{ij} = -2\Psi\eta_{ij}. \ (14)$$

As a result, the scalar curvature and 00 component of the Ricci curvature tensor around the flat Minkowski background are given by

$$R = 4\Delta\Psi(r) - 2\Delta\Phi(r), \qquad R_{00} = \Delta\Phi(r), \quad (15)$$

where $\Delta \equiv \eta^{ij}\partial_i\partial_j$ is the Laplace operator.

Then from the equations of motion (2), the trace and 00 component equations of motion are [5]

$$-\kappa\rho = \kappa T = \frac{1}{2}\left(a(\Box) - 3c(\Box)\right)R,$$

$$\kappa\rho = \kappa T_{00} = a(\Box)R_{00} + \frac{1}{2}c(\Box)R. \qquad (16)$$

Here we have taken the weak field approximation so that $\rho \gg p$, where ρ is the density and p is the pressure of the test mass. Therefore $T = (-\rho + 3p) \approx -\rho$ and $T_{00} \approx \rho$. Combining this with (15) leads to

$$\Delta\Psi(r) = -\frac{c(\Box)}{(a(\Box) - 2c(\Box))}\Delta\Phi(r),$$

$$\Delta\Phi(r) = \frac{a(\Box) - 2c(\Box)}{a(\Box)\left(a(\Box) - 3c(\Box)\right)}\kappa\rho. \qquad (17)$$

As our point source is of mass μ, its density is approximated by a 3-dimensional Dirac-delta function: $\rho = \mu\,\delta^3(\boldsymbol{r})$. Next we use a Fourier transform to calculate $\Phi(r)$ by using the same method as for calculating the Coulomb potential [43, 44].

When the Dirac-delta function is Fourier transformed into momentum space or p-space, it becomes [15]

$$\rho = \mu\,\delta^3(\boldsymbol{r}) = \mu\int \frac{d^3p}{(2\pi)^3}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}. \qquad (18)$$

As we go into momentum space, we take $\Box \to -p^2$, giving

$$\Phi(r) = -\frac{\kappa\mu}{(2\pi)^3}\int_{-\infty}^{\infty}d^3p\frac{a-2c}{a(a-3c)}\frac{e^{i\boldsymbol{p}\cdot\boldsymbol{r}}}{p^2}$$

$$= -\frac{\kappa\mu}{4\pi^2 r}\int_{-\infty}^{\infty}dk\frac{(a-2c)}{a(a-3c)}\frac{\sin(pr)}{p}. \qquad (19)$$

Trivially, we can use (17) to see that

$$\Psi(r) = \frac{\kappa\mu}{4\pi^2 r}\int_{-\infty}^{\infty}dp\frac{c}{a(a-3c)}\frac{\sin(pr)}{p}. \qquad (20)$$

Note that if we set $a(-p^2) = c(-p^2)$,

$$\Phi(r) = \Psi(r) = -\frac{\kappa\mu}{4\pi^2 r}\int_{-\infty}^{\infty}dp\frac{1}{2a}\frac{\sin(pr)}{p}. \qquad (21)$$

This case was discussed in [15, 31]. For the simplest case, where $a(\Box) = e^{-\Box/M^2}$, it results in the $1/r$ fall of the potential seen in GR being modified by the error function,

i.e. $\Phi(r) \sim \frac{\text{Erf}(r)}{r}$. For large r, $\text{Erf}(r) \approx 1$ but at short distances, $\text{Erf}(r) \sim r$, so $\Phi(r) \sim 1$. Therefore the $1/r$ behaviour is retained at large distances but at short distances, the modification means that the potential simply tails off to a constant and the potential is no longer singular.

A. Potential for IDG with defocusing

In this paper we will extend the calculation of the IDG potential to the case where $a \neq c$, which allows us to avoid singularities. We describe the relationship between $a(\Box)$ and $c(\Box)$ using (10), and by inserting (10) into (19) we find that

$$\Phi(r) = -\frac{\kappa\mu}{24\pi^2 r}$$

$$\times \int_{-\infty}^{\infty}dp\left[4 - \frac{m^2}{\tilde{a}(-p^2)(m^2 + p^2)}\right]\frac{\sin(pr)}{p\,a(-p^2)}. \quad (22)$$

The calculation we need to perform is

$$f(r) = \int_{-\infty}^{\infty}dp\left[4 - \frac{m^2}{\tilde{a}(-p^2)(m^2 + p^2)}\right]\frac{\sin(pr)}{p\,a(-p^2)}, \quad (23)$$

where $\Phi(r) \sim -\frac{f(r)}{r}$. We can write $\tilde{a}(-p^2) = e^{\tau(-p^2)}$ and $a(-p^2) = e^{\gamma(-p^2)}$, and this results in

$$f(r) = \int_{-\infty}^{\infty}dp\left[4 - \frac{m^2 e^{-\tau(-p^2)}}{m^2 + p^2}\right]\frac{e^{-\gamma(-p^2)}\sin(pr)}{p}. \quad (24)$$

This is our main result - we have shown that we can have defocusing as well as a non-singular Newtonian potential. This potential returns to the GR value in the infrared limit, i.e for large values of r.[6]

B. Conditions on $a(-p^2)$ and $\tilde{a}(-p^2)$

Next we investigate the conditions that must be placed on $a(-p^2)$ and $\tilde{a}(-p^2)$, and therefore what we can say about $\tau(-p^2)$ and $\gamma(-p^2)$.

First we look at the spin-0 part of the propagator. From (11), we have

$$\Pi(p^2) = \frac{1}{e^{\gamma(-p^2)}}\left[\frac{P^{(2)}}{p^2} + \frac{P^{(0)}}{2p^2(1 + p^2/m^2)e^{\tau(-p^2)}}\right], \quad (25)$$

where $P^{(2)}$ is the spin-2 projection operator and $P^{(0)}$ is the spin-0 projection operator. We require that in the UV, the propagator is exponentially suppressed, allowing us the possibility of curtailing divergences [28]. From the spin-0 sector of (25), we observe the condition $\gamma(-p^2) +$

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[5] We have noted that the time derivatives of $R(r)$ and $R_{00}(r)$ vanish.

[6] Note that in the limit $M \to \infty$ we return to a non-local theory, which is fourth-order gravity [45].

$\tau(-p^2) > 0$. Fortunately, this condition ensures that the integral (24) converges. Note that a priori, it is possible that there are coefficients in front of the exponentials $e^{\gamma(-p^2)}$ and $e^{\tau(-p^2)}$, but by requiring that we return to the GR propagator in the infrared limit $k \ll m, M$, we are obliged to set the coefficients to 1. Also, by requiring the spin-2 part of the propagator (25) to be exponentially suppressed we have $\gamma(-p^2) > 0$.

From looking at (12) in momentum space, the defocusing condition imposes a constraint: $\tau(-p^2) \geq 0$. Thus, we have the functions $a(-p^2) = e^{\gamma(-p^2)}$ and $\tilde{a}(-p^2) = e^{\tau(-p^2)}$ where $\gamma(-p^2)$ is positive function and $\tau(-p^2)$ is a non-negative function. In fact, the result of these constraints is that the integral (24) converges.

We therefore have two free mass scales M and m and two unspecified functions $\tau(-p^2)$ and $\gamma(-p^2)$. We will henceforth choose both the mass scales to be the Planck mass M_P and choose the simplest possible version of $\tau(-p^2)$, which is $\tau = 0$. This choice means that all of the freedom in the model is tied up in the function $\gamma(-p^2)$, which we will choose to be the monomial $\gamma(p^2) = (Cp^2/M_P^2)^D$. We therefore have only two free parameters, C and D.

FIG. 2: We plot $f(r)$ vs r for different D and $C = 1$ where $\gamma(p^2) = (Cp^2/M_P^2)^D$. See (26). The plot shows that for $D > 1$ the potential oscillates. As the value of D increases, so too does the magnitude of these oscillations. For illustrative purposes, we have taken $M_p = 1\mathrm{m}^{-1}$ and again we find that our results reduce to that of GR at large distances.

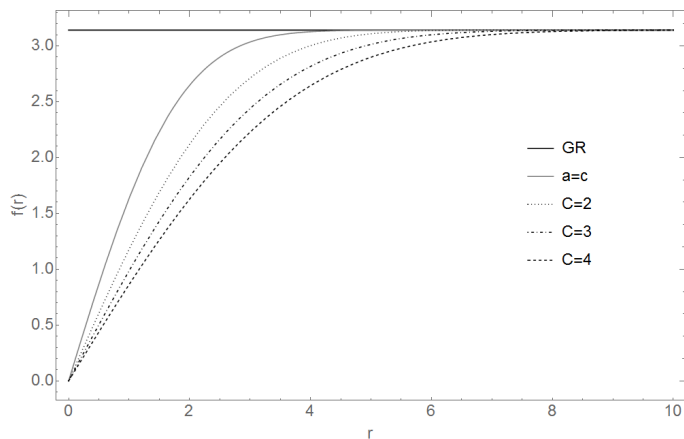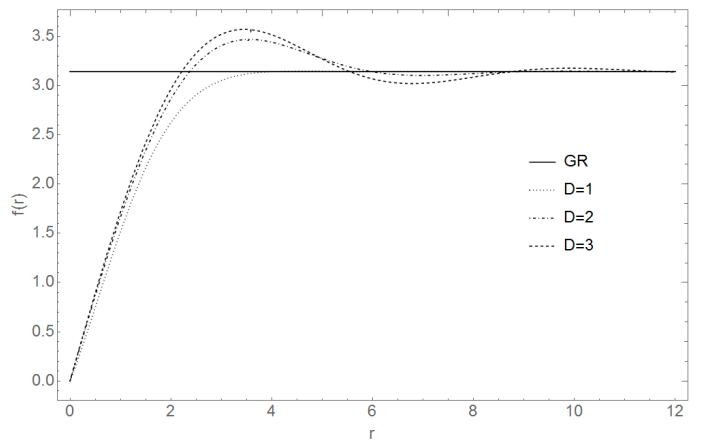## IV. PLOTTING THE RESULTS

FIG. 1: We plot $f(r)$ vs r for different C and $D = 1$, where $\gamma(p^2) = (Cp^2/M_P^2)^D$. See (26). For illustrative purposes, we have taken $M_p = 1\mathrm{m}^{-1}$. We compare our results to the $a(\Box) = c(\Box)$ case seen in (21). Here, we see that as C increases, the effect of IDG can be seen increasingly further away from the origin.
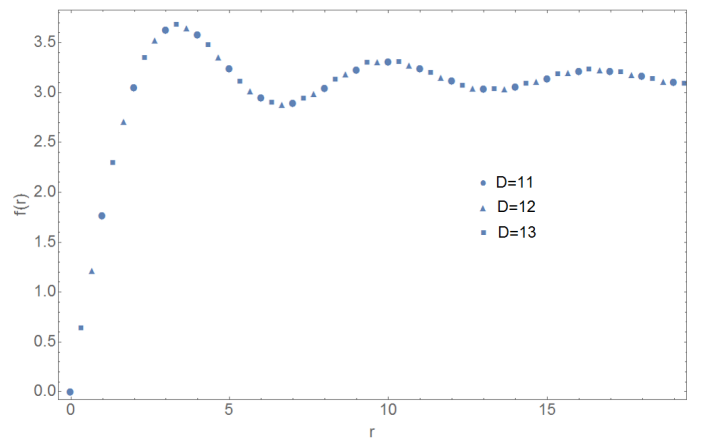

FIG. 3: We plot $f(r)$ vs r for different D greater than 10 with $C = 1$, where $\gamma(p^2) = (Cp^2/M_P^2)^D$. See (26). We can see that for $D > 10$, increasing D does not affect the potential. For illustrative purposes, we have taken $M_p = 1\mathrm{m}^{-1}$. We can parameterise these curves as $\alpha_1 r$ for $r < 1$ and $\alpha_2 \cos(\theta r + \theta_0)/r$ for $r > 1$, where α_1, α_2, θ and θ_0 are constants, as in [38].

A. Choosing a form for $a(-p^2)$ and $\tilde{a}(-p^2)$

For the simplest choice $\tau(-p^2) = 0$ and $\gamma(p^2) = (Cp^2/M_P^2)^D$, this gives[7]

$$f(r) = \int_{-\infty}^{\infty} dp \left[4 - \frac{M_P^2}{M_P^2 + p^2} \right] \frac{e^{-(Cp^2/M_P^2)^D} \sin(pr)}{p}. \quad (26)$$

In Fig. 1, we take $D = 1$ and plot $f(r)$ for different choices of C using (26). Clearly increasing the value of C moves the point at which the non-locality kicks in further away from the origin. Next in Fig. 2 we look at how the potential varies for the same C but with different values of D, which is the power of p^2 in (26).

As we increase the value of D, our potential begins to oscillate, as was found in [31]. These oscillations grow in size as D increases until about $D = 10$.

This is because

$$\lim_{D \to \infty} e^{-(Cp^2/M_P^2)^D} = \mathrm{rect}(Cp^2/M_P^2), \quad (27)$$

where $\mathrm{rect}(x)$ is the rectangle function, which is defined by $\mathrm{rect}(x) = 1$ for $|x| < 1$ and $\mathrm{rect}(x) = 0$ for $|x| > 1$. For $D > 10$, (27) is a very good approximation and so increasing the value of D does not change the potential.

Our next task is to investigate the choice of large powers of p^2 which has been shown to fit recent experimental data [38] at small distances for the $a(\Box) = c(\Box)$ case. We will see whether we can still obtain oscillating solutions with conditions necessary to realise defocusing. In Fig. 3 we plot (26) for different choices of $D > 10$ and note that it can still be parameterised accurately as

$$f(r) = \begin{cases} \alpha_1 r & \text{for } 0 < r < 1, \\ 1 + \alpha_2 \frac{\cos(\theta r + \theta_0)}{r} & \text{for } 1 < r. \end{cases}$$

In other words, the oscillating solution which was hinted at by experimental data can be produced by a modified gravity solution which also allows geodesic completeness.

V. CONCLUSION

We have found the Newtonian potential for a wider class of Infinite Derivative Gravity (IDG) theories than were previously investigated and analysed various cases of the theory. Analysis of data from experimental tests has hinted that, at small distances, an oscillating non-local potential provides the best fit to experimental data. We have shown that an IDG theory constrained to allow the defocusing of null rays, and therefore geodesic completeness,

still produces a non-singular potential which returns to the standard GR result at large distances. This result can still be parameterised as the oscillating function which provides a good fit to the data.

By allowing defocusing, it is necessary to introduce extra parameters into the model, although we can reduce the freedom in these parameters by making appropriate choices about the mass scales and the form of the functions.

Our results can be tested experimentally, which will allow us to put constraints on our parameters. Future research could look at moving away from the minimal model and reintroducing the choice of parameters which we reduced, or finding the potential around a de Sitter background rather than a flat space background.

VI. ACKNOWLEDGEMENTS

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